Projective theory of meson fields and electromagnetic properties of atomic nuckei

PROJECTIVE THEORY OF MESON FIELDS AND ELECTROMAGNETIC PROPERTIES OF ATOMIC NUCLEI

BY

A. PAIS

PROJECTIVE THEORY OF MESON FIELDS AND ELECTROMAGNETIC PROPERTIES OF ATOMIC NUCLEI

Diss. Utrecht 1941

PROJECTIVE THEORY OF MESON FIELDS AND ELECTROMAGNETIC PROPERTIES OF ATOMIC NUCLEI

PROEFSCHRIFT

TER VERKRIJGING VAN DEN GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DEN RECTOR MAGNIFICUS DR. H. R. KRUYT, HOOGLEERAAR IN DE FACULTEIT DER WIS- EN NATUURKUNDE, VOLGENS BESLUIT VAN DEN SENAAT DER UNIVERSITEIT TEGEN DE BEDENKINGEN VAN DE FACULTEIT DER WIS- EN NATUURKUNDE TE VERDEDIGEN OP WOENSDAG 9 JULI 1941, DES NAMIDDAGS TE 3 UUR

DOOR

ABRAHAM PAIS

GEBOREN TE AMSTERDAM

AMSTERDAM — 1941

N.V. NOORD-HOLLANDSCHE UITGEVERS MAATSCHAPPIJ

AAN VADER EN MOEDER.
AAN TINEKE.

(Op verzoek van den promotor, prof. dr. L. ROSENFELD volgt hier, in plaats van het gebruikelijke voorwoord, een korte levensschets.)

19 Mei 1918 werd ik te Amsterdam geboren. Daar bezocht ik de derde 5-jarige H.B.S. aan de Mauritskade, (directeur dr. GERRITS), waar ik veel heb opgestoken. In 1935 legde ik het eindexamen af, en liet me in hetzelfde jaar als student aan de Amsterdamse Gemeente Universiteit inschrijven. Oorspronkelijk koos ik de fysisch-chemische richting (e), maar veranderde al spoedig van koers, vooral onder de invloed van de colleges van prof. MANNOURY. Tenslotte besloot ik, na vele gesprekken met enige oudere jaars, de theoretisch-fysische kant op te gaan.
Voor het candidaatsexamen volgde ik de colleges van prof. CLAY, prof. MANNOURY, (zoals gezegd), prof. MICHELS, prof. PANNEKOEK, prof. HK. DE VRIES, (ik denk nog steeds met plezier aan het caput over meetkunde van het aantal), prof. WIBAUT en dr. BÜCHNER. Op 16 Februari 1938 legde ik het candidaatsexamen *a* en *d* af. In dat jaar volgde ik nog enige wiskunde-colleges van prof. BROUWER, prof. WEITZENBÖCK en dr. FREUDENTHAL.
Het was in deze tijd, dat ik in aanraking gekomen ben met en opgenomen in een kringetje van mensen, die ik het beste zou kunnen karakteriseren door onze eigenschap om alles te kunnen lachen, vooral om onszelf. In tweeërlei opzicht is dit contact beslissend voor me geweest, (vooral in het tweede).
In het voorjaar van 1938 ging ik naar Utrecht, om bij prof. UHLENBECK theoretische natuurkunde te studeren. Veel heb ik geleerd van zijn glasheldere colleges en de colloquia in kleine kring op „220"; maar vooral de tijd gedurende welke ik met hem aan enige theoretische problemen heb mogen werken, is een mooie tijd voor me geweest.

Met grote eerbied herdenk ik hier prof. ORNSTEIN. In de tijd dat ik experimenteel werkte op het Utrechts laboratorium heb ik hem kunnen waarnemen in die hoedanigheid, waarin hij zo groot was: als organisator van een brok fysisch leven. De bijna dagelijkse gesprekken die ik met hem had in het jaar na het vertrek van prof. UHLENBECK zijn een kostbare herinnering voor me.
En nog iemand wil ik hier gedenken: KEES VAN LIER.

Voor mijn doctoraal examen volgde ik nog de colleges van prof. BARRAU en prof. WOLFF; dit examen legde ik op 22 April 1940 af. In die cursus heb ik met dankbaarheid gebruik gemaakt van de gastvrijheid, mij door prof. KRAMERS op het theoretisch seminarium te Leiden geboden.
In September 1940 kwam prof. ROSENFELD naar Utrecht en onder zijn leiding heb ik, oorspronkelijk als zijn assistent, mijn studie voortgezet. Ik beschouw het als een voorrecht, dat ik onder zijn toezicht dit proefschrift heb kunnen bewerken. Zijn steun, zijn aansporingen en vooral de belangstelling die hij steeds in mijn

persoon stelde hebben me door menig moeilijk moment heengeholpen. Het is vooral van zo grote waarde voor me dat ik in hem iemand gevonden heb, die me niet alleen veel op het gebied van de fysica geleerd heeft, (en nog veel zal leren, naar ik hoop), maar met wien ik tegelijkertijd ook zoveel contact heb kunnen vinden op andere gebieden. De hartelijkheid, die hij en zijn vrouw mij steeds betoond hebben, is een grote steun voor me geweest in een moeilijke tijd.

Ik betreur het, dat door de omstandigheden de in deze dissertatie behandelde problemen op sommige punten niet zo behandeld zijn, als dat in mijn voornemen lag. Ik hoop evenwel later op deze kwesties terug te komen.

Aan hen, die mij het dierbaarst zijn, draag ik dit proefschrift op.

CHAPTER I.

THE ENERGY MOMENTUM TENSOR IN PROJECTIVE RELATIVITY THEORY.

Summary.

After a survey of the formalism of projective relativity theory (§ 2) an expression is derived for the 5-dimensional energy momentum tensor of an arbitrary field (§ 3). It is proved that, in virtue of the equations which hold for the variables describing the field, this tensor is symmetrical and its divergence vanishes. This last property expresses the conservation of energy, momentum and charge of the system. As an application of the formalism the energy momentum tensor for the Dirac field is computed in § 4.

§ 1. *Introduction.* Since KALUZA [1]) in 1921 pointed out that the unification of the gravitational and electromagnetic field might be achieved by introducing a fifth dimension besides space-time of general relativity, several attempts have been made, starting from this idea, to obtain a formalism which satisfies the following requirements:

a. General covariance, (covariant formulation of the "cylinder-condition").

b. The first set of Maxwell equations follows from the postulated structure of 5-dimensional space.

c. The field equations are derivable from a variational principle.

d. The "geodesic" equations of 5-dimensional space represent the equations of motion of a charged mass point in the gravitational *and* electromagnetic field.

The method of KALUZA which was extended and improved by O. KLEIN [2]) does not fulfill the first condition, as it starts from a 5-dimensional metrical tensor, the components of which do not depend on the fifth coordinate, and which is in fact nothing but the ordinary metrical tensor bordered by the electromagnetic vector potential, while g_{55} is put equal to 1. EINSTEIN and MAYER [3]) have proposed another method, *viz.* of adjoining a linear 5-dimensional vector space to every point of the 4-dimensional space-time continuum. In this case, however, the conditions b and c are not fulfilled.

The projective interpretation which was first introduced by VEBLEN and HOFFMANN [4]) considers the 5-dimensional space as a 4-dimensional projective one. The treatment of these authors exhibits the same defect as that of the KALUZA–KLEIN theory *), but in the projective formalism ultimately developed by VAN DANTZIG, SCHOUTEN [5]) and PAULI [6]) the covariance requirement is indeed satisfied.

In the theory of SCHOUTEN and VAN DANTZIG the three-index symbols $\Gamma^{\lambda}_{\mu\nu}$ of projective space are not symmetrical with respect to μ and ν. PAULI has emphasized however, that there is no physical argument for not keeping to symmetrical $\Gamma^{\lambda}_{\mu\nu}$. Starting from this assumption one then gets a uniquely determined formalism. The metrical tensor $g_{\mu\nu}$ is assumed to fulfill the (covariant) condition

$$g_{\mu\nu} X^{\mu} X^{\nu} = 1.$$

The last mentioned authors have also discussed the DIRAC theory of the electron in the frame of this formalism and have derived an expression for the energy momentum tensor of the DIRAC field. The aim of the present paper is to do this for an arbitrary field of which the Lagrange function is given in terms of the field variables.

The problem of the derivation of the energy momentum tensor in general relativity theory has recently been discussed by BELINFANTE [7]) and by ROSENFELD [8]). These treatments differ methodically and the first mentioned has for our purposes the drawback that the differential conservation laws are only derived in the approximation of special relativity. As it seems that no simple physical meaning can be attached to a "special projective relativity theory" it will be clear that it is more convenient for us to proceed on the lines of ROSENFELD's method which follows closely the ideas outlined especially by HILBERT [9]).

It should be noted that the situation here differs in two aspects from that in general relativity: first, it is here no more allowed to assume that the Lagrangian does not depend explicitly on the coordinates and secondly, the transformation group of projective rela-

---

*) The projective tensor $\gamma_{\alpha\beta}$ introduced in equ. (3. 1) *loc. cit.* [4]) has essentially the same properties as the metrical tensor of the KALUZA–KLEIN theory.

tivity (h_5), in contrast to the group of general relativity (g_4), admits only such transformations $X^\mu \rightarrow X'^\mu$ for which X'^μ is a *homogeneous function of the first degree* of the X^μ.

§ 2. *Survey of the formalism of projective relativity theory.* We shall first summarize the main features of projective relativity theory according to PAULI's formalism. For the omitted proofs and a more detailed treatment of the subject, the reader is referred to PAULI's paper.

a) *5-tensors and 5-tensor densities.* The space-time continuum is alternatively described by the inhomogeneous coordinates x^i (to this description we will refer as "4-space") or by the homogeneous coordinates *) X^μ (5-space), such that $x^1, \ldots, x^4$ are homogeneous functions of zeroth degree in X^μ:

$$x^i = x^i(X^1, \ldots, X^5) = x^i(\varrho X^1, \ldots, \varrho X^5). \qquad (1)$$

We now define the 5-tensor $T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}$ as a set of quantities which obey the following transformation law for the group h_5:

$$T'^{\bar{\alpha}_1 \ldots \bar{\alpha}_n}_{\bar{\beta}_1 \ldots \bar{\beta}_k} = \frac{\partial X'^{\bar{\alpha}_1}}{\partial X^{\alpha_1}} \cdots \frac{\partial X'^{\bar{\alpha}_n}}{\partial X^{\alpha_n}} \frac{\partial X^{\beta_1}}{\partial X'^{\bar{\beta}_1}} \cdots \frac{\partial X^{\beta_k}}{\partial X'^{\bar{\beta}_k}} T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}; \qquad (2)$$

further, each tensor component shall satisfy the invariant condition that it be a homogeneous function of degree p of X^μ, where p is given by:

$$p = n - k.$$

Therefore, using Euler's theorem on homogeneous functions and denoting by $A^{\cdots}_{\cdots|\mu}$ the partial derivative of $A^{\cdots}_{\cdots}$ with respect to X^μ:

$$X^\mu T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k|\mu} = p\, T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}. \qquad (3)$$

Thus the X^μ are components of a contravariant 5-vector field, while the differentials dX^μ do not have this property as the second condition is not fulfilled for them. Still one can define

$$d\tau = dX^1 \ldots dX^5$$

*) We will adhere to the convention that 4-space quantities are marked by Latin and 5-space quantities by Greek indices, so $i = 1 \ldots 4$, and $\mu = 1 \ldots 5$. Summation signs are suppressed according to the usual rule.

as volume-element in 5-space: if we perform the transformation

$$X'^{\mu} = \varrho X^{\mu}, \tag{4}$$

(ϱ any homogeneous function of the X^{μ} of degree zero), $d\tau$ transforms according to *)

$$d\tau' = \varrho^5 \, d\tau. \tag{5}$$

As a consequence of (4) we also have

$$T'^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k} = \varrho^p \, T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}. \tag{6}$$

This enables us to find the degree of 5-tensor density components. For, if $t^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}$ is a 5-tensor density and $T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}$ the corresponding 5-tensor integral:

$$T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k} = \int t^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k} \, d\tau,$$

we see that the degree of $t^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}$ is $p - 5$, because of (5) and (6) **).

Noting that $X^{\mu}{}_{|\mu} = 5$ we therefore have:

$$\left(t^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k} X^{\mu} \right)_{|\mu} = p \, t^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_k}. \tag{7}$$

b) *Metric; covariant differentiation.* The metric is described by the symmetrical tensor $g_{\mu\nu}$:

$$g_{\mu\nu} = g_{\nu\mu}. \tag{8}$$

Further we postulate the following important relation, normalizing the metric in 5-space

$$g_{\mu\nu} X^{\mu} X^{\nu} = 1; \tag{9}$$

this relation is invariant for the group h_5. The raising and lowering of indices can be performed by introducing the tensor $g^{\mu\nu}$ which is connected in the usual way with $g_{\mu\nu}$.

Next the three index symbols $\Gamma^{\lambda}_{\mu\nu}$ and $\Gamma_{\lambda,\mu\nu}$ are introduced, which we assume to be symmetrical in μ and ν:

$$\Gamma_{\lambda,\mu\nu} = \tfrac{1}{2} \left(g_{\lambda\mu|\nu} + g_{\lambda\nu|\mu} - g_{\mu\nu|\lambda} \right) \tag{10a}$$

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\sigma} \, \Gamma_{\sigma,\mu\nu}. \tag{10b}$$

---

*) Comp. PAULI, *loc. cit.* [6]) p. 311.

**) It is well known that the square root of the absolute value of Det $g_{\mu\nu}$, (where $g_{\mu\nu}$ is the metrical tensor introduced in *b* below), is a scalar density. Thus its degree is -5, as is also easily verified, the degree of $g_{\mu\nu}$ being -2.

It is well known that, while $T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k|\mu}$, the derivative of a mixed tensor with respect to X^μ is not a mixed tensor of rank $p-1$, (except when T is a scalar), on the other hand the covariant derivative of $T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k}$:

$$T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k||\mu} = T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k|\mu} + \sum_{i=1}^{n} \Gamma^{\alpha_i}_{\mu\lambda}\, T^{\alpha_1\ldots\alpha_{i-1},\lambda,\alpha_{i+1}\ldots\alpha_n}_{\beta_1\ldots\beta_k} - \\ - \sum_{i=1}^{k} \Gamma^{\lambda}_{\mu\beta_i}\, T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_{i-1},\lambda,\beta_{i+1}\ldots\beta_k}$$

does possess this property. Also we remind that for covariant differentiation the product rule holds:

$$(A^{\cdots}_{\cdots}\, B^{\cdots}_{\cdots})_{||\mu} = A^{\cdots}_{\cdots||\mu}\, B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots}\, B^{\cdots}_{\cdots||\mu},$$

and that the operations of covariant differentiation and raising or lowering of indices are commutable on account of

$$g_{\lambda\mu||\nu} = 0, \quad g^{\lambda\mu}{}_{||\nu} = 0. \tag{11}$$

Finally we introduce the 5-tensor $X_{\mu\nu}$:

$$X_{\mu\nu} = X_{\nu||\mu} - X_{\mu||\nu} = X_{\nu|\mu} - X_{\mu|\nu} = -X_{\nu\mu}, \tag{12}$$

which satisfies

$$X_{\mu\nu|\varrho} + X_{\varrho\mu|\nu} + X_{\nu\varrho|\mu} = 0. \tag{12a}$$

From (3) and (10*a*) we infer that

$$X^\lambda\, \Gamma_{\nu,\mu\lambda} = -g_{\mu\nu} + \tfrac{1}{2} X_{\mu\nu}, \tag{13}$$

so

$$X^\nu{}_{|\mu} = \tfrac{1}{2} X_\mu^{\cdot\nu}, \quad X_{\nu||\mu} = \tfrac{1}{2} X_{\mu\nu}. \tag{13a}$$

With the help of (9) one verifies that

$$X_{\mu\nu}\, X^\nu = 0. \tag{14}$$

c) Connection between 5-tensors and 4-tensors *). These relations are established by the 40 quantities $\gamma_\nu^{\cdot k}$ and $\gamma^\nu{}_{.k}$:

$$\gamma_\nu^{\cdot k} \equiv x^k{}_{|\nu} \tag{15}$$

*) A 5-scalar is at the same time a 4-scalar, therefore it is not necessary to add a prefix to "scalar".

whence, by (1)

$$\gamma_{\nu}^{\cdot k} X^{\nu} = 0; \tag{16}$$

$\gamma^{\nu}_{\cdot k}$ is defined by

$$\begin{cases} \gamma_{\nu}^{\cdot k} \gamma^{\nu}_{\cdot i} = \delta_{i}^{k} & (17) \\ \gamma^{\nu}_{\cdot i} X_{\nu} = 0. & (18) \end{cases}$$

$\gamma_{\nu}^{\cdot k}$ transforms like a covariant 5-vector for the group h_5, and like a contravariant 4-vector for g_4; $\gamma^{\nu}_{\cdot k}$ behaves similarly.

With the help of $\gamma_{\nu}^{\cdot k}$ $(\gamma^{\nu}_{\cdot k})$ we can uniquely connect a contravariant (covariant) 4-vector with a contravariant (covariant) 5-vector by

$$a^k = \gamma_{\nu}^{\cdot k} a^{\nu} \quad ; \quad a_k = \gamma^{\nu}_{\cdot k} a_{\nu}. \tag{19a}$$

From (19a) it follows that the metrical 4-tensor, which connects a_i and a^k, is related to $g_{\mu\nu}$ by

$$g_{ik} = \gamma^{\mu}_{\cdot i} \gamma^{\nu}_{\cdot k} g_{\mu\nu}. \tag{21a}$$

Furthermore, (17), (18) and (9) give

$$\gamma^{\mu}_{\cdot i} \gamma_{\nu}^{\cdot i} = \delta^{\mu}_{\cdot \nu} - X_{\nu} X^{\mu},$$

and this relation enables us to connect (starting from (19a)) a contravariant (covariant) 5-vector with a contravariant (covariant) 4-vector. For, contracting *) the first equation (19a) with $\gamma_{\mu}^{\cdot k}$ and the second with $\gamma^{\mu}_{\cdot k}$ we get

$$a^{\mu} = \gamma^{\mu}_{\cdot k} a^k + a X^{\mu} \quad ; \quad a_{\mu} = \gamma_{\mu}^{\cdot k} a_k + a X_{\mu} \tag{19b}$$
$$a = a_{\mu} X^{\mu} = a^{\mu} X_{\mu}.$$

There is a special type of 5-vectors, namely those for which the corresponding scalar vanishes: $a_{\mu} X^{\mu} = 0$. Such vectors we denote by $\bar{a}_{\mu}$, (or $\bar{a}^{\mu}$).

In the same way connections can be established between 5-tensors and 4-tensors of higher rank. For instance

$$T_{ik} = \gamma^{\mu}_{\cdot i} \gamma^{\nu}_{\cdot k} T_{\mu\nu}, \tag{20a}$$

$$T_{\mu\nu} = \gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} T_{ik} + \gamma_{\mu}^{\cdot i} X_{\nu} T_{i(0)} + X_{\mu} \gamma_{\nu}^{\cdot i} T_{(0)i} + T_{(0)(0)}. \tag{20b}$$

*) By contraction we understand multiplication followed by summation over the new dummy index (indices).

with

$T_{i(0)} = \gamma^{\mu}_{.i} X^{\nu} T_{\mu\nu}$, $T_{(0)i} = X^{\mu} \gamma^{\nu}_{.i} T_{\mu\nu}$ and $T_{(0)(0)} = X^{\mu} X^{\nu} T_{\mu\nu}$.

So we see that

if $T_{\mu\nu} = T_{\nu\mu}$, then $T_{ik} = T_{ki}$, $T_{i(0)} = T_{(0)i}$;

if $T_{\mu\nu} = - T_{\nu\mu}$, then $T_{ik} = - T_{ki}$, $T_{i(0)} = - T_{(0)i}$, $T_{(0)(0)} = 0$.

From (20*b*) and (18) it is seen that we especially have for the metrical 5-tensor:

$$g_{\mu\nu} = \gamma_{\mu}^{.i} \gamma_{\nu}^{.k} g_{ik} + X_{\mu} X_{\nu}. \tag{21b}$$

Now we still have to find the relations between the covariant derivative of a 5-tensor in 5-space with the analogous quantity in 4-space. For this purpose we postulate, for any vector $\bar{a}^{\nu}$ (i.e. a vector for which $\bar{a}^{\nu} = \gamma^{\nu}_{.k} a^k$ holds, see above):

$$a^k{}_{||l} = \gamma^{\varrho}_{.l} \gamma_{\nu}^{.k} \bar{a}^{\nu}{}_{||\varrho},$$

where $a^k{}_{||l}$ denotes the ordinary covariant derivative in 4-space. Next we define $\gamma_{\nu}^{.k}{}_{||\mu}$ and $\gamma^{\nu}_{.k||\mu}$ as follows

$$\gamma_{\nu}^{.k}{}_{||\mu} = \gamma_{\nu}^{.k}{}_{|\mu} - \Gamma^{\lambda}_{\nu\mu} \gamma_{\lambda}^{.k} + \overset{4}{\Gamma}{}^{k}_{lm} \gamma_{\mu}^{.m} \gamma_{\nu}^{.l}, \tag{22a}$$

$$\gamma^{\nu}_{.k||\mu} = \gamma^{\nu}_{.k|\mu} + \Gamma^{\nu}_{\lambda\mu} \gamma^{\lambda}_{.k} - \overset{4}{\Gamma}{}^{l}_{km} \gamma_{\mu}^{.m} \gamma^{\nu}_{.l}, \tag{22b}$$

where $\overset{4}{\Gamma}{}^{k}_{lm}$ is the three-index symbol in 4-space. On account of these definitions the following product-rules hold

$$(\gamma_{\nu}^{.k} a_k)_{||\varrho} = \gamma_{\nu}^{.k} \gamma_{\varrho}^{.l} a_{k||l} + \gamma_{\nu}^{.k}{}_{||\varrho} a_k,$$

$$(\gamma^{\nu}_{.k} a^k)_{||\varrho} = \gamma^{\nu}_{.k} \gamma_{\varrho}^{.l} a^k{}_{||l} + \gamma^{\nu}_{.k||\varrho} a^k.$$

Furthermore, it can be seen that the postulate then becomes

$$\gamma_{\nu}^{.m} \gamma^{\varrho}_{.n} \gamma^{\nu}_{.k||\varrho} = 0.$$

We now proceed to derive explicit expressions for the covariant derivative of $\gamma_{\nu}^{.k}$ and $\gamma^{\nu}_{.k}$.

We have, by (19*b*) and (13*a*)

$$(\gamma^{\nu}_{.k} a^k)_{||\varrho} = a^{\nu}{}_{||\varrho} - X^{\nu} a_{||\varrho} - \tfrac{1}{2} a X_{\varrho}^{.\nu}.$$

Therefore, using (3), (13), (14) and (16):

$$X^{\varrho}\,\gamma^{\nu}_{.k\|\varrho} = -\tfrac{1}{2}\,X^{\nu}_{.\varrho}\,\gamma^{\varrho}_{.k}.$$

Also we have

$$X_{\nu}\,\gamma^{\nu}_{.k\|\varrho} = -\gamma^{\nu}_{.k}\,X_{\nu\|\varrho} = -\tfrac{1}{2}\,X_{\varrho\nu}\,\gamma^{\nu}_{.k}.$$

Using a relation similar to (20*b*) which holds for a mixed 5-tensor of the second rank, we thus get

$$\gamma^{\nu}_{.k\|\varrho} = \tfrac{1}{2}\,(X_{\varrho}\,X_{\mu}^{.\nu} - X^{\nu}\,X_{\varrho\mu})\,\gamma^{\mu}_{.k}\ . \tag{23a}$$

Likewise we find that

$$\gamma_{\nu}^{.k}{}_{\|\varrho} = \tfrac{1}{2}\,(-X_{\varrho}\,X_{\nu}^{.\mu} + X_{\nu}\,X^{\mu}_{.\varrho})\,\gamma_{\mu}^{.k}\ . \tag{23b}$$

d) The postulate dealing with the electromagnetic field tensor. Introducing

$$X_{ik} = \gamma^{\mu}_{.i}\,\gamma^{\nu}_{.k}\,X_{\mu\nu}, \tag{24}$$

we have (comp. (20*b*) and (14)):

$$X_{\mu\nu} = \gamma_{\mu}^{.i}\,\gamma_{\nu}^{.k}\,X_{ik}. \tag{25}$$

With the help of (12), (33) and (34) one can prove that *)

$$X_{ik|l} + X_{li|k} + X_{kl|i} = 0. \tag{26}$$

So if we put X_{ik} proportional to the electromagnetic field F_{ik}, we see that the first set of Maxwell equations is a consequence of the assumed structure of 5-space. The proportionality-factor can be determined by comparing the field equations of gravitation theory (in the absence of matter) combined with those of electrodynamics on the one hand with the corresponding equations that follow from the projective formalism on the other; it turns out to be $(2\varkappa)^{1/2}\,c^{-1}$ where $\varkappa$ is the gravitational constant, thus

$$X_{ik} = \frac{\sqrt{2\varkappa}}{c}\,F_{ik}. \tag{27}$$

From (26) it follows that there exists a 4-vector f_i, such that:

$$X_{ik} = f_{k|i} - f_{i|k}. \tag{26a}$$

---

*) Comp. PAULI [6]), p. 322.

By (27) f_i is connected with the electromagnetic vector potential φ_i by

$$f_k = \frac{\sqrt{2\varkappa}}{c}\,\varphi_k. \tag{28}$$

We may connect f_k with a 5-vector $\bar{f}_\mu$:

$$\bar{f}_\mu = \gamma_{\mu}^{\cdot k} f_k;\ (\bar{f}_\mu X^\mu = 0),$$

it can be proved *) that

$$\bar{f}_\mu = X_\mu - \frac{1}{F} F_{|\mu}, \tag{29}$$

where F is an undetermined homogeneous function of the *first* degree of X^μ.

e) *Connection between* $g_{\mu\nu}$ *and the* DIRAC *matrices.* The linearization of the GORDON–SCHRÖDINGER equation of the electron rests essentially upon the existence of 4 square matrices γ_i which can have no less than 4 rows and columns and which satisfy

$$\gamma_i \gamma_k + \gamma_k \gamma_i = 2\delta_{ik}.$$

The product of those matrices: $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ is also anticommutative with each γ_i, while $\gamma_5^2 = 1$. These matrix relations can all be comprised in

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu},\ (\mu, \nu = 1, \ldots, 5) \tag{30}$$

where $\delta_{\mu\nu}$ is the unit matrix. Now just as TETRODE [10]) postulated that the matrices γ_i should be generalized to matrix fields describing gravitation in 4-space, we can put here for a given metrical field $g_{\mu\nu}$, (writing a_μ instead of γ_μ):

$$a_\mu a_\nu + a_\nu a_\mu = 2 g_{\mu\nu}. \tag{31}$$

For given $g_{\mu\nu}$ it is in principle possible to find a solution for a_μ (comp. SCHRÖDINGER [11]), which is unique apart from a transformation with any non-singular matrix S:

$$a'_\mu = S^{-1} a_\mu S,\ (\text{or } a'_\mu = -S^{-1} a_\mu S). \tag{32}$$

*) Cf. PAULI, p. 322—323.

For reasons that will soon be clear we consider only S-transformations of the first type.

Further there exists a matrix A, the "hermitizing matrix" such that

$$A\,\alpha_\mu = (A\,\alpha_\mu)^\dagger. \tag{33}$$

As a consequence of (32) A is assumed to transform as follows:

$$A' = S^\dagger A S. \tag{34}$$

f) 5-undors. We call

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$$

a 5-undor if, performing (32), it transforms according to

$$\Psi' = S^{-1}\,\Psi. \tag{35}$$

From (34) and (35) we obtain

$$(\Psi^\dagger A)' = (\Psi^\dagger A)\,S.$$

The quantities $\Psi^\dagger A\,\Psi$, $\Psi^\dagger A\,\alpha_\mu\,\Psi$, $\Psi^\dagger A\,\alpha_\mu\alpha_\nu\,\Psi$ etc. are therefore invariant for S-transformations.

Next we consider a rotation of the X^μ:

$$X'^\mu = \eta^\mu_{.\nu}\,X^\nu, \tag{36}$$

($\eta^\mu_{.\nu}$ is homogeneous of degree zero in X^μ), and we will confine ourselves to rotations with Det $\eta^\mu_{.\nu} = +1$. The consideration of this group, of which the full group of LORENTZ transformations (including spatial reflections) is an undergroup, is sufficient for all physical purposes.

Now it is always possible to find a matrix Σ, such that if (with $\alpha^\mu = g^{\mu\nu}\,\alpha_\nu$)

$$\alpha'^\mu = \eta^\mu_{.\nu}\,\alpha^\nu, \tag{37}$$

then

$$\alpha'^\mu = \Sigma^{-1}\,\alpha^\mu\,\Sigma. \tag{38}$$

Such a "Σ-transformation" only affects the α^μ, the quantities Ψ, $\Psi^\dagger$ and A are scalars with respect to these. Therefore $\Psi^\dagger A \Psi$ etc. mentioned above are real scalars, 5-vectors, etc. with respect to the transformations (37).

It is possible to connect uniquely a S-transformation with a Σ-transformation by stating that a S-transformation and the "adjoined" Σ-transformation, which shall be performed subsequently, shall leave the α^μ unchanged. We then have in fact $S = \Sigma^{-1}$ (with respect to the choice we have made as regards (32)).

§ 3. *The energy momentum tensor.* We now proceed to derive an expression for the energy momentum tensor of an arbitrary field. We generally denote the variables describing the field by $Q_{(\omega)}$. This symbol thus comprises the gravitational field variables (to which we will often refer as $Q_{(\gamma)}$) and the others ($Q_{(\alpha)}$) which have either tensor or undor character, (we denote them with $Q_{(\tau)}$ and $Q_{(\sigma)}$ respectively). We will, only to fix our thoughts, consider the $Q_{(\alpha)}$'s to be covariant tensor components. If in a term the "index" (ω), (α), (τ) or (σ) occurs twice, "summation" over all variables $Q_{(\omega)}$ or $Q_{(\alpha)}$, etc. is implied.

We will — in contrast to BELINFANTE [7]) and ROSENFELD [8]) — not establish the connection with undors by the explicit introduction of "5-beinchen" but by the direct consideration of a set of matrices α_μ varying from point to point and satisfying (31).

In case we only have to do with variables of the type $Q_{(\tau)}$ (the "tensor case") we adhere to the customary choice of the $Q_{(\gamma)}$, namely the components of the metrical tensor $g_{\mu\nu}$. If there are also undor variables present, (the "general case"), this choice cannot be maintained; we now take the α_μ instead, connected with the $g_{\mu\nu}$ by (31).

We denote by K the Lagrangian density of the gravitational 5-field in the absence of matter, (K is assumed to depend on $Q_{(\gamma)}$ and their first derivatives only), and by L the Lagrangian density of the arbitrary material field, containing the interaction of that field with the $g_{\mu\nu}$-field, (this is the interaction with gravitation *and* the electromagnetic field). L depends on $Q_{(\omega)}$, their first derivatives, and on X^μ:

$$L = L(Q_{(\omega)}, Q_{(\omega)|\mu}, X^\mu). \tag{39}$$

The explicit dependence of L on the coordinates is an essential feature of the projective formalism which finds no counterpart in general relativity theory.

We obtain the gravitational equations in 5-space by putting:

$$\delta \int (K + L)\, d\tau = 0; \tag{40}$$

the variation of the integral should be taken for arbitrary variations of independent combinations of the $Q_{(\gamma)}$. The integration is to be extended over the domain of 5-space corresponding with the domain of 4-space occupied by the system, the independent combinations of the $Q_{(\gamma)}$ vanishing on the border.

Therefore we have in the tensor case, (taking into account (8) and (9)):

$$\int \left\{ \frac{\delta(K+L)}{\delta g_{\mu\nu}}\, \delta g_{\mu\nu} + \eta\, (\delta g_{\mu\nu} - \delta g_{\nu\mu}) + \lambda X^{\mu} X^{\nu}\, \delta g_{\mu\nu} \right\} d\tau = 0, \tag{41a}$$

η and λ being Lagrange multiplicators; the quantities $\delta K/\delta g_{\mu\nu}$ etc. are the variational derivatives of K with respect to $g_{\mu\nu}$, etc. (41a) gives

$$K^{\mu\nu} - K^{(0)} X^{\mu} X^{\nu} = T^{\mu\nu} - T^{(0)} X^{\mu} X^{\nu} \tag{41b}$$

with

$$K^{\mu\nu} = \frac{\delta K}{\delta g_{\mu\nu}} + \frac{\delta K}{\delta g_{\nu\mu}},$$

$$T^{\mu\nu} = -\left(\frac{\delta L}{\delta g_{\mu\nu}} + \frac{\delta L}{\delta g_{\nu\mu}} \right), \tag{42a}$$

$$K^{(0)} = K^{\mu\nu} X_{\mu} X_{\nu}\ ,\ T^{(0)} = T^{\mu\nu} X_{\mu} X_{\nu}.$$

Here $T_{\mu\nu}$ is the energy momentum 5-tensor.

In the general case, (42*a*) is extended, making use of well known properties of variational derivatives and of (31), to

$$T^{\mu\nu} = -\frac{1}{2} \left(a^{\mu} \frac{\delta L}{\delta a_{\nu}} + \frac{\delta L}{\delta a_{\nu}}\, a^{\mu} \right). \tag{42b}$$

In the following we will for this and analogous formulae more simply write

$$T^{\mu\nu} = -\frac{\delta L}{\delta a_\nu} a^\mu,$$

the symmetrization being understood; we may of course assume the a's to be hermitized.

The invariance properties for the group h_5 of the quantities hitherto introduced enable us to find an explicite expression for $T_{\mu\nu}$ in terms of the field-variables. Consider an infinitesimal transformation

$$X'^\mu = X^\mu + \xi^\mu, \tag{43}$$

(ξ^μ an infinitesimal contravariant 5-vector); we then have by (3):

$$X^\nu \xi^\mu{}_{|\nu} = \xi^\mu, \tag{44}$$

As a consequence of (43), the field variables transform as follows

$$\delta Q_{(\alpha)} = Q'_{(\alpha)}(X'^\mu) - Q_{(\alpha)}(X^\mu) = c^\mu_{(\omega),\nu}\, \xi^\nu{}_{|\mu}, \tag{45}$$

where

$$c^\mu_{(\omega),\nu} = -\sum_{k=1}^{p} \delta^\mu_{\nu_k}\, Q_{(\omega)\,\nu_1 \ldots \nu_{k-1},\, \mu,\, \nu_{k+1} \ldots \nu_p}, \tag{46}$$

$\nu_1, \ldots, \nu_p$ are the tensor indices of $Q^{(\omega)}$.

Comparing this with (42*a*) and (42*b*) one infers that

$$T^{\mu}_{.\nu} = \frac{\delta L}{\delta Q_{(\gamma)}}\, c^\mu_{(\gamma),\nu}. \tag{42c}$$

It will appear that it is convenient to introduce the "local variation" $\delta^* Q_{(\omega)} = Q'_{(\omega)}(X^\mu) - Q_{(\omega)}(X^\mu)$. Thus:

$$\delta Q_{(\omega)} = \delta^* Q_{(\omega)} + \xi^\mu Q_{(\omega)\,|\mu}.$$

The second part of the right member is the result of the concomitant displacement $Q'_{(\omega)}(X^\mu) \rightarrow Q'_{(\omega)}(X'^\mu)$.

Now $\int L d\tau$ should be invariant for the group h_5, i.e. we have *):

$$\delta \int L\, d\tau = \int [\delta^* L + (L \xi^\mu)_{|\mu}]\, d\tau = 0, \tag{47}$$

---

*) Cf. H. WEYL, Raum, Zeit, Materie, 5nd ed., p. 233; it should be noted that in computing $\delta^* L$ we need not vary X^μ as $\delta^* L$ is the *local* variation of L.

where ξ^μ and its derivatives must satisfy (44) and all relations that can be obtained from it by differentiation.

On account of (39) the variational derivative of L with respect to $Q_{(\omega)}$ is

$$\frac{\delta L}{\delta Q_{(\omega)}} = \frac{\partial L}{\partial Q_{(\omega)}} - \left(\frac{\partial L}{\partial Q_{(\omega)\,|\,\mu}}\right)_{|\mu}, \tag{48}$$

so (47) becomes

$$\int d\tau \,[A_\nu \xi^\nu + (B^{\mu}_{.\nu}\, \xi^\nu + R^{\mu\lambda}_{\nu}\, \xi^\nu{}_{|\lambda})_{|\mu}] = 0, \tag{49}$$

where we have introduced the following abbreviations

$$A_\nu = -\left(\frac{\delta L}{\delta Q_{(\omega)}}\, c^{\mu}_{(\omega),\,\nu}\right)_{|\mu} - \frac{\delta L}{\delta Q_{(\omega)}}\, Q_{(\omega)\,|\,\nu}, \tag{50}$$

$$B^{\mu}_{.\nu} = L\delta^{\mu}_{.\nu} - \frac{\partial L}{\partial Q_{(\omega)\,|\,\mu}}\, Q_{(\omega)\,|\,\nu} + \frac{\delta L}{\delta Q_{(\omega)}}\, c^{\mu}_{(\omega),\,\nu}, \tag{51}$$

$$R^{\mu\lambda}_{\nu} = \frac{\partial L}{\partial Q_{(\omega)\,|\,\mu}}\, c^{\lambda}_{(\omega),\,\nu}. \tag{52}$$

If we first consider an infinitesimal *linear* transformation of X^μ, the second derivatives of ξ^μ do no more occur in (49). We then obtain from (49) using (44):

$$(A_\nu + B^{\varrho}_{.\nu|\varrho})\, X^\mu + B^{\mu}_{.\nu} + R^{\lambda\mu}_{\nu\,|\lambda} = 0. \tag{53}$$

For a general infinitesimal transformation the condition

$$\int R^{\mu\lambda}_{\nu}\, \xi^\nu{}_{|\lambda|\mu} = 0$$

remains, where the $\xi^\nu{}_{|\lambda|\mu}$ are only restricted *) by the derivative of (44), *viz.*

$$X^\mu \xi^\nu{}_{|\lambda|\mu} = 0, \tag{54}$$

*) Higher derivatives of (44) need not be taken into account, (although the derivative of (54) still contains $\xi^\nu{}_{\lambda|\mu}$). For, by counting the number of equations with which such a tensor relation is aequivalent on the one hand and the number of quantities $\xi^\nu{}_{|\lambda|\mu}$, $\xi^\nu{}_{|\lambda|\mu|\varrho}$ etc., that are involved in such a set of equations on the other, it is easily seen that the higher derivatives of (43) do not impose any further restriction on ξ_μ and their first and second derivatives.

whence

$$R_{\nu}^{(\lambda\mu)} = \Omega_{\nu}^{(\lambda} X^{\mu)}; \qquad R_{\nu}^{(\lambda\mu)} = R_{\nu}^{\lambda\mu} + R_{\nu}^{\mu\lambda}. \tag{55a}$$

The wholly undetermined quantity Ω_{ν}^{λ} which, on account of (54), has to be introduced into (55), expresses an ambiguity of the Lagrange function which is typical for the projective formalism; in fact we may always replace $Q_{(\omega)}$ by a constant times $X^{\lambda} Q_{(\omega)|\lambda}$ (on account of (3) if $Q_{(\omega)}$ is a tensor; for a 5-undor we have e.g. $X^{\lambda} \Psi_{|\lambda} = l\Psi$, see later on equ. (97)), but if we do so $R_{\nu}^{\lambda\mu}$ changes, according to its definition (52), by an amount $\triangle R_{\nu}^{\lambda\mu}$, where

$$\triangle R_{\nu}^{\lambda\mu} = \beta^{(\omega)} \frac{\partial L}{\partial Q_{(\omega)}} c_{(\omega),\nu}^{\mu} X^{\lambda},$$

($\beta^{(\omega)}$ is the constant mentioned above; of course one should *not* perform the summation over (ω) in this expression).

Noting that, naturally, $c_{(\omega),\nu}^{\mu}$ cannot be written as the product of a 5-tensor "$c_{(\omega)\nu}$" with X^{μ}, we consequently may, if we introduce the convention not to allow products of the type $X^{\lambda} Q_{(\omega)|\lambda}$ to occur in L, put Ω_{ν}^{λ} equal to zero:

$$R_{\nu}^{(\lambda\mu)} = 0; \tag{55b}$$

thus $R_{\nu}^{\lambda\mu}$ is antisymmetrical in λ and μ from which follows that $R_{\nu|\lambda|\mu}^{\lambda\mu} = 0$. We therefore obtain from (53) by differentiation:

$$\{(A_{\nu} + B_{.\nu|\varrho}^{\varrho}) X^{\mu}\}_{|\mu} + B_{.\nu|\mu}^{\mu} = 0.$$

As $A_{\nu} + B_{.\nu|\varrho}^{\varrho}$ is 5-vector density we have by (7)

$$\{(A_{\nu} + B_{.\nu|\varrho}^{\varrho}) X^{\mu}\}_{|\mu} = -A_{\nu} - B_{.\nu|\varrho}^{\varrho},$$

whence, using (50)

$$-A_{\nu} = \left(\frac{\delta L}{\delta Q_{(\omega)}} c_{(\omega),\nu}^{\mu}\right)_{|\mu} + \frac{\delta L}{\delta Q_{(\omega)}} Q_{(\omega)|\nu} = 0. \tag{56}$$

Inserting this into (51) we get

$$B_{.\nu|\mu}^{\mu} = L_{|\nu} - \left(\frac{\partial L}{\partial Q_{(\omega)|\mu}} Q_{(\omega)|\nu}\right)_{|\mu} - \frac{\delta L}{\delta Q_{(\omega)}} Q_{(\omega)|\nu}.$$

Therefore, from (48) and (39) it follows that

$$B_{.\nu|\mu}^{\mu} = \frac{\partial^e L}{\partial^e X^{\nu}}, \tag{57}$$

where the right member is defined as the derivative of L with respect to those X^ν which occur explicitly.

Inserting (57) in (54) and taking account of (56) we get:

$$L\,\delta^\mu_\nu - \frac{\partial L}{\partial Q_{(\omega)|\mu}}\,Q_{(\omega)|\nu} + \frac{\delta L}{\delta Q_{(\omega)}}\,c^\mu_{(\omega),\nu} + R^{\lambda\mu}_{\nu|\lambda} + X^\mu\frac{\partial^e L}{\partial^e X^\nu} = 0,$$

therefore (see (42c)):

$$T^\mu_{.\nu} = \frac{\partial L}{\partial Q_{(\omega)|\mu}}\,Q_{(\omega)|\nu} - L\,\delta^\mu_\nu - R^{\lambda\mu}_{\nu|\lambda} - X^\mu\,\frac{\partial^e L}{\partial^e X^\nu} - \frac{\delta L}{\delta Q_{(\alpha)}}\,c^\mu_{(\alpha)\nu}. \quad (58)$$

As $T^\mu_{.\nu}$ only depends on the *derivative* of $R^{\lambda\mu}_\nu$, it follows that the occurrence of Ω^λ_ν would not affect the energy momentum tensor, for

$$(\Omega^\nu_\mu\,X^\lambda)_{|\lambda} = 0,$$

on account of (7), Ω^ν_μ being a tensor density.

Now we must bring (56) and (58) in their covariant form. This is easily brought about in the tensor case, for then we have, noting that $\delta L/\delta Q_{(\omega)}\,.\,c^\mu_{(\omega),\nu}$ and $R^{\lambda\mu}_\nu$ are tensor densities:

$$\left(\frac{\delta L}{\delta Q_{(\omega)}}\,c^\mu_{(\omega),\nu}\right)_{||\mu} = \left(\frac{\delta L}{\delta Q_{(\omega)}}\,c^\mu_{(\omega),\nu}\right)_{|\mu} - \frac{\delta L}{\delta Q_{(\omega)}}\,c^\mu_{(\omega),\lambda}\,\Gamma^\lambda_{\mu\nu},$$

$$R^{\lambda\mu}_{\nu||\lambda} = R^{\lambda\mu}_{\nu|\lambda} - R^{\lambda\mu}_\varrho\,\Gamma^\varrho_{\lambda\nu}.$$

So, as

$$Q_{(\tau)||\mu} = Q_{(\tau)|\mu} + c^\lambda_{(\tau),\varrho}\,\Gamma^\varrho_{\lambda\mu}, \quad (59)$$

we have, using (11):

$$T^\mu_{.\nu} = \frac{\partial L}{\partial Q_{(\tau)||\mu}}\,Q_{(\tau)||\nu} - L\,\delta^\mu_\nu - R^{\lambda\mu}_{\nu||\lambda} - X^\mu\,\frac{\partial^e L}{\partial^e X^\nu} - \frac{\delta L}{\delta Q_{(\tau)}}\,c^\mu_{(\tau),\nu}\,,$$

$$T^\mu_{.\nu||\mu} + \left(\frac{\delta L}{\delta Q_{(\tau)}}\,c^\mu_{(\tau),\nu}\right)_{||\mu} + \frac{\delta L}{\delta Q_{(\tau)}}\,Q_{(\tau)||\nu} = 0.$$

In the general case we may write instead of these last two identities

$$T^\mu_{.\nu} = \frac{\partial L}{\partial a^\varrho_{;\mu}}\,a^\varrho_{;\nu} + \frac{\partial L}{\partial Q_{(\alpha)||\mu}}\,Q_{(\alpha);\nu} - L\,\delta^\mu_\nu - R^{\lambda\mu}_{\nu||\lambda} - X^\mu\,\frac{\partial^e L}{\partial^e X^\nu} - \frac{\delta L}{\delta Q_{(\alpha)}}\,c^\mu_{(\alpha),\nu}\,. \quad (60)$$

$$T^{\mu}_{.\nu||\mu} + \left(\frac{\delta L}{\delta Q_{(\alpha)}} c^{\mu}_{(\alpha),\nu}\right)_{||\mu} + \frac{\delta L}{\delta a^{\mu}} a^{\mu}{}_{;\nu} + \frac{\delta L}{\delta Q_{(\alpha)}} Q_{(\alpha);\nu} = 0, \quad (61)$$

with

$$Q_{(\alpha);\mu} = Q_{(\alpha)|\mu} + c^{\lambda}_{(\alpha)\varrho} \Gamma^{\varrho}_{\lambda\mu}. \quad (59a)$$

Thus $Q_{(\tau);\mu} \equiv Q_{(\tau)||\mu}$, but the covariant derivative of a tensor undor [13]) is not completely given by an expression like (59*a*). In this case we can achieve our purpose by using the invariance of $\int L d\tau$ for a change in representation corresponding with a Σ-transformation in the sense as indicated above *). Such an (infinitesimal) transformation is given by **):

$$\begin{cases} \delta' a^{\mu} = \varepsilon^{\mu\nu} a_{\nu}, \\ \delta' g_{\mu\nu} = 0, \end{cases} \quad (62)$$

from which follows

$$\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}.$$

We can then bring $\delta' a^{\mu}$ into the form

$$\delta' a^{\mu} = d_{(\alpha)}{}^{\mu}_{\lambda\nu} \varepsilon^{\lambda\nu}, \quad (63)$$

with

$$d_{(\alpha)}{}^{\mu}_{\lambda\nu} = \tfrac{1}{2}(\delta^{\mu}_{\lambda} a_{\nu} - \delta^{\mu}_{\nu} a_{\lambda}). \quad (63a)$$

As a consequence of (62) the variables $Q_{(\sigma)}$ are also affected:

$$\delta' Q_{(\sigma)} = d_{(\sigma)\lambda\nu} \varepsilon^{\lambda\nu}. \quad (64)$$

It should be noted that $\delta' A = 0$, for Aa'^{μ} is hermitian because Aa^{μ} is hermitian and the transformation coefficients $\varepsilon^{\mu\nu}$ are real. We now assume that L depends in such a way on $Q_{(\sigma)}$, $Q_{(\sigma)|\varrho}$, a^{μ} and $a^{\mu}{}_{|\varrho}$ that the (vanishing) variation of $\int L d\tau$ can be written as follows

$$\delta' \int L\, d\tau = \int \left(\frac{\delta L}{\delta a^{\mu}} \delta' a^{\mu} + \frac{\delta L}{\delta Q_{(\sigma)}} \delta' Q_{(\sigma)}\right) + \\ + \int \left(\frac{\partial L}{\partial a^{\mu}{}_{|\nu}} \delta' a^{\mu} + \frac{\partial L}{\partial Q_{(\sigma)|\nu}} \delta' Q_{(\sigma)}\right)_{|\nu} = 0, \quad (65)$$

*) See PAULI, p. 350.

**) We have written δ' to distinguish these variations from those that follow from (43).

where, as already stated, all quantities in the second member of (65) stand for their symmetrized expressions, i.e. one has actually to take half the sum of the quantity written down and its hermitian conjugate. From (65) the following identities can be derived

$$\frac{\delta L}{\delta a^{\mu}} d_{(\alpha)\lambda\nu}^{\ \ \mu} + \frac{\delta L}{\delta Q_{(\sigma)}} d_{(\sigma)\lambda\nu} = 0, \tag{66}$$

$$\frac{\partial L}{\partial a^{\mu}{}_{|\varrho}} d_{(\alpha)\lambda\nu}^{\ \ \mu} + \frac{\partial L}{\partial Q_{(\sigma)|\varrho}} d_{(\sigma)\lambda\nu} = 0. \tag{67}$$

Now with any parallel displacement ξ'^{μ} in 5-space (from P to Q say) corresponds a change of representation, namely $\alpha^{\mu}(P) \rightarrow \alpha^{\mu}(Q)$, with a $\delta'\alpha^{\mu}$ of the form (62), and it is this connection which causes the difference between $Q_{(\sigma)||\mu}$ and $Q_{(\sigma);\varrho}$. From

$$\delta' \alpha^{\mu} = - \alpha^{\mu}{}_{;\varrho}\, \xi'^{\varrho} \tag{68}$$

we can readily deduce the quantities $\varepsilon^{\mu\nu}$ corresponding with ξ'^{μ}. For, as the α_{μ} are linear combinations of a set of constant matrices $\mathring{\alpha}_{\mu}$, (see PAULI, p. 344), we have

$$\alpha^{\mu}{}_{;\varrho} = \triangle^{\mu\nu}_{\cdot\cdot\varrho}\, \alpha_{\nu}, \tag{69}$$

where $\triangle^{\mu\nu}_{\cdot\cdot\varrho}$ is a c-number. One can easily deduce from (68) that the $\varepsilon^{\mu\nu}$ are connected with ξ'^{ϱ} by

$$\varepsilon^{\mu\nu} = Y^{\mu\nu}_{\cdot\cdot\varrho}\, \xi'^{\varrho}, \tag{70}$$

where

$$Y^{\mu\nu}_{\cdot\cdot\varrho} = - \tfrac{1}{2} (\triangle^{\mu\nu}_{\cdot\cdot\varrho} - \triangle^{\nu\mu}_{\cdot\cdot\varrho}). \tag{70a}$$

Consequently we get for the covariant derivative of $Q_{(\sigma)}$ and of α^{μ}:

$$Q_{(\sigma)||\varrho} = Q_{\sigma;\varrho} + Y^{\lambda\nu}_{\cdot\cdot\varrho}\, d_{(\sigma)\lambda\nu}, \tag{71}$$

$$\alpha^{\mu}{}_{||\varrho} = \alpha^{\mu}{}_{;\varrho} + Y^{\lambda\nu}_{\cdot\cdot\varrho}\, d_{(\alpha)\lambda\nu}^{\ \ \mu} = 0. \tag{72}$$

(From (72) and (11) it follows that $\alpha_{\mu||\varrho} = 0$). Thus, contracting (66) and (67) with $Y^{\lambda\nu}_{\cdot\cdot\varrho}$ we get

$$\frac{\delta L}{\delta \alpha^{\mu}} \alpha^{\mu}{}_{;\varrho} + \frac{\delta L}{\delta Q_{(\sigma)}} Q_{(\sigma);\varrho} = \frac{\delta L}{\delta Q_{(\sigma)}} Q_{(\sigma)||\varrho},$$

$$\frac{\partial L}{\partial \alpha^{\lambda}{}_{|\mu}} \alpha^{\lambda}{}_{;\nu} + \frac{\partial L}{\partial Q_{(\sigma)|\mu}} Q_{(\sigma);\nu} = \frac{\partial L}{\partial Q_{(\sigma)||\mu}} Q_{(\sigma)||\nu},$$

which is exactly what we need to obtain the covariant form of (60) and (61); for, inserting this we find

$$T^{\mu}_{.\nu} = \frac{\partial L}{\partial Q_{(\alpha)\,||\mu}} Q_{(\alpha)\,||\nu} - L\delta^{\mu}_{\nu} - R^{\lambda\mu}_{\nu\,|\lambda} - X^{\mu} \frac{\partial^e L}{\partial^e X^{\nu}} - \frac{\delta L}{\delta Q_{(\alpha)}} c^{\mu}_{(\alpha),\nu}, \quad (73)$$

$$T^{\mu}_{.\nu||\mu} + \left(\frac{\delta L}{\delta Q_{(\alpha)}} c^{\mu}_{(\alpha),\nu}\right)_{||\mu} + \frac{\delta L}{\delta Q_{(\alpha)}} Q_{(\alpha)\,||\nu} = 0. \quad (74)$$

Naturally we must require that in the right hand side of (73) the gravitational quantities do not occur explicitly and indeed this condition is fulfilled except for $R^{\lambda\mu}_{\nu}$ which, by its definition (52), contains $\partial L/\partial Q_{(\gamma)\,|\lambda} \cdot c^{\mu}_{(\gamma),\nu}$.

In order to eliminate this term from $R^{\lambda\mu}_{\nu}$ we put

$$s^{\mu\nu}_{(\omega)} = \tfrac{1}{2}\,(c^{\mu}_{(\omega)\,\lambda}\, g^{\lambda\nu} - c^{\nu}_{(\omega)\,\lambda}\, g^{\lambda\mu}) + d^{\mu\nu}_{(\omega)}, \quad (75)$$

with $d^{\mu\nu}_{(\gamma)} = d^{\mu\nu}_{(\tau)} \equiv 0$, while $d^{\mu\nu}_{(\sigma)}$ is given by (64). For $s^{\mu\nu}_{(\gamma)}$ we have thus in the tensor case $s^{\mu\nu}_{(\gamma)} = 0$, while in the general case, according to (63a), we have for $Q_{(\gamma)} \equiv a_{\varrho}$,

$$s^{\mu\nu}_{(\gamma)} = -\, d_{(\alpha)\,\varrho}{}^{\mu\nu} \equiv - g_{\varrho\varrho'}\, g^{\mu\mu'}\, g^{\nu\nu'}\, d_{(\alpha)}{}^{\varrho'}_{\mu'\nu'}. \quad (75a)$$

We now introduce

$$D^{\lambda;\,\mu\nu} = \tfrac{1}{2}\,(R^{\lambda\mu}_{\varrho}\, g^{\varrho\nu} - R^{\lambda\nu}_{\varrho}\, g^{\varrho\mu}),$$

and then can write, using the identity (67), and (52), (75) and (75a)

$$D^{\lambda;\,\mu\nu} = \frac{\partial L}{\partial Q_{(\alpha)\,||\lambda}}\, s^{\mu\nu}_{(\alpha)}. \quad (76)$$

Thus

$$R^{\lambda\mu}_{\nu} = g_{\nu\varrho}\,[D^{\lambda;\,\mu\varrho} - D^{\varrho;\,\lambda\mu} + D^{\mu;\,\varrho\lambda}], \quad (77)$$

and we see that in this form $R^{\lambda\mu}_{\nu}$ does no more contain $Q_{(\gamma)}$ explicitly.

The symmetry of $T^{\mu\nu}$ and the conservation laws. Representing the energy momentum tensor in its form (42b) we have from (66):

$$T^{\mu\nu} - T^{\nu\mu} = 2\,\frac{\delta L}{\delta Q_{(\sigma)}}\, d_{(\sigma)}{}^{\mu\nu}. \quad (78)$$

If we form from the variables $Q_{(\alpha)}$ a set of independent variables $Q_{(\bar{\alpha})}$ *), the field equations are:

$$\frac{\delta L}{\delta Q_{(\bar{\alpha})}} = 0, \tag{79}$$

consequently

$$\frac{\delta L}{\delta Q_{(\alpha)}} c^{\mu}_{(\alpha),\nu} = 0,$$

which enables us to modify (78) to

$$\bar{T}^{\mu}_{.\nu} = \frac{\partial L}{\partial Q_{(\alpha)||\mu}} Q_{(\alpha)||\nu} - L\,\delta^{\mu}_{\nu} - R^{\lambda\mu}_{\nu||\lambda} - X^{\mu}\frac{\partial^e L}{\partial^e X^{\nu}} \tag{80}$$

(we have written $\bar{T}^{\mu}_{.\nu}$ to distinguish this quantity from the corresponding tensor in (58)). As regards the symmetry of $\bar{T}^{\mu}_{.\nu}$, on account of (42*b*), (78) and (75)

$$\bar{T}^{\mu\nu} - \bar{T}^{\nu\mu} = 2\,\frac{\delta L}{\delta Q_{(\alpha)}} s^{\mu\nu}_{(\alpha)}.$$

The right hand side vanishes however, if we make use of the field equations (79). Therefore $\bar{T}^{\mu\nu}$ is symmetrical *in virtue of these equations*. As from (79) it also follows that

$$\frac{\delta L}{\delta Q_{(\alpha)}} Q_{(\alpha)||\nu} = 0,$$

the conservation laws hold under the same restriction:

$$\bar{T}^{\mu}_{.\nu||\mu} = 0. \tag{81}$$

Dividing (80) and (81) by $\sqrt{g}$, ($g = |\mathrm{Det}\, g_{\mu\nu}|$), we obtain instead of tensor densities the corresponding tensors. Then (81) is, on account of (20*b*) and (23), aequivalent with

$$\bar{T}^{ik}{}_{||k} - X^{ik}\,\bar{T}_{k\,(0)} = 0,$$

$$\bar{T}^{k}_{.(0)||k} = 0.$$

Putting

$$\bar{T}^{ik} = -\frac{\varkappa}{c^2}\,T^{ik}, \tag{82a}$$

---

*) We assume $Q_{(\bar{\alpha})}$ to be a homogeneous and linear function of the $Q_{(\alpha)}$'s.

$$\overline{T}_{k(0)} = -\frac{\sqrt{\varkappa}}{c\sqrt{2}} s_k, \tag{82b}$$

where T_{ik} is the energy momentum 4-tensor of the total system *minus* the tensor referring to the Maxwell field in the absence of matter: $T_{(e)ik}$, and s_k is the charge current density of the system we have, with the help of (27):

$$T^{ik}{}_{||k} - F^{ik} s_k = 0\,, \tag{83}$$

$$\sqrt{\overline{g}}\,.\, s^k{}_{||k} = (\sqrt{\overline{g}}\, s^k)_{|k} = 0\,. \tag{84}$$

$$\overline{g} = |\mathrm{Det.}\, g_{ik}|.$$

So the 5 identities (81) are aequivalent with the conservation of momentum, energy and charge!

From (41*b*) we now derive two four-dimensional relations by contracting with $\gamma_{\mu}^{\cdot i}\,\gamma_{\nu}^{\cdot k}$ and $X_{\mu}\,\gamma_{\nu}^{\cdot i}$ respectively; they are

$$K_{ik} = -\frac{\varkappa}{c^2} T_{ik}, \tag{85a}$$

$$K^{i}_{\cdot(0)} = \overline{T}^{i}_{\cdot(0)}. \tag{85b}$$

The left members have been computed by PAULI, who finds

$$K_{ik} = R_{ik} - \tfrac{1}{2} g_{ik} R + \frac{\varkappa}{c^2} (F_i{}^l F_{lk} - \tfrac{1}{4} g_{ik} F_{mn} F^{mn}),$$

$$K^{i}_{\cdot(0)} = -\frac{\sqrt{\varkappa}}{c\sqrt{2}} F^{ik}{}_{||k}\,;$$

R_{ik} is the contracted RIEMANN-tensor.

Thus (85*a*) is aequivalent with the equations of EINSTEIN's gravitation theory, while (85*b*) becomes

$$F^{ik}{}_{||k} = s^i, \quad \text{or} \quad \frac{1}{\sqrt{\overline{g}}} (\sqrt{\overline{g}}\, F^{ik})_{|k} = s^i, \tag{86}$$

i.e. the second set of Maxwell equations.

§ 4. DIRAC *theory in projective form.* In case that $Q_{(\sigma)}$ is the 5-undor Ψ we have (see (64), (75) and PAULI [6]), p. 351):

$$s^{\mu\nu} = d^{\mu\nu} = -\tfrac{1}{8}\, a^{[\mu\nu]}\, \Psi, \quad a^{[\mu\nu]} = a^{\mu} a^{\nu} - a^{\nu} a^{\mu}. \tag{87}$$

We now introduce Λ_ϱ:

$$\Lambda_\varrho = -\tfrac{1}{8}\, a_{[\mu\nu]}\, Y^{\mu\nu}_{\cdot\cdot\varrho} = -\tfrac{1}{4}\, a_\mu\, a_\nu\, Y^{\mu\nu}_{\cdot\cdot\varrho}\,. \tag{88}$$

Thus, on account of (71), $\Psi_{||\varrho} = \Psi_{|\varrho} + \Lambda_\varrho\, \Psi$. We are however still free to replace Λ_ϱ by a quantity Γ_ϱ differing from Λ_ϱ by a multiple of the unit matrix. The choice which is most adapted to our purposes is

$$\Gamma_\varrho = \Lambda_\varrho - l X_\varrho, \tag{88}$$

with *)

$$l = \frac{ie}{hc} \cdot \frac{c}{\sqrt{2\varkappa}}. \tag{89a}$$

We then have

$$\Psi_{||\varrho} = \Psi_{|\varrho} + \Lambda_\varrho\, \Psi - l X_\varrho\, \Psi, \tag{90a}$$

similarly

$$\Psi^\dagger_{||\varrho} = \Psi^\dagger{}_{|\varrho} - \Psi^\dagger\, \Lambda_\varrho + l X_\varrho\, \Psi. \tag{90b}$$

The expression for these covariant derivatives here chosen differ from those of PAULI by the last term.

Using

$$a^\mu\, a_\lambda\, a_\nu - a_\lambda\, a_\nu\, a^\mu = 2\,(\delta^\mu_\lambda\, a_\nu - \delta^\mu_\nu\, a_\lambda),$$

it is easily seen that (72) can be brought into the form

$$a^\mu{}_{||\varrho} = a^\mu{}_{;\varrho} + \Lambda_\varrho\, a^\mu - a^\mu \Lambda_\varrho = 0. \tag{72a}$$

As regards the covariant derivative of A, we can normalize it in such a way that **)

$$A_{||\mu} = 0. \tag{91}$$

We now introduce the DIRAC-equation in 5-space:

$$a^\mu\, \Psi_{||\mu} + \eta\, \Psi = 0, \qquad \eta = \frac{imc}{h}. \tag{92}$$

This equation can be derived from the following Lagrangian ***)

$$L = Re\, \frac{hc}{i}\, (\Psi^\dagger A a^\mu\, \Psi_{||\mu} + \eta\, \Psi^\dagger A\, \Psi).\, \sqrt{g}\,. \tag{93}$$

---

*) h is PLANCK's constant divided by 2π.

**) See PAULI [6]), p. 359.

***) In (93) and the following formulae we have looked apart from the factor $-c^2/\varkappa$ (cf. (82)).

We can now compute the energy momentum tensor. In doing so we must, according to our assumption as regards the derivation of (65), consider the Λ_ϱ's as functions of α^μ and $\alpha^\mu{}_{;\varrho}$. This is possible, as a solution for Λ_ϱ satisfying (72*a*) is

$$\Lambda_\varrho = \tfrac{1}{8}\,(\alpha_{\mu;\varrho}\,\alpha^\mu - \alpha^\mu\,\alpha_{\mu;\varrho}),$$

as can be verified, using (69).

Using further (87) and the analogous relation holding for $\Psi^\dagger$:

$$s^{\mu\nu} = -\tfrac{1}{8}\,\Psi^\dagger\,\alpha^{[\mu\nu]\dagger},$$

we find from (76)

$$D^{\lambda;\mu\nu} = Re - \frac{hc}{8\,i}\,\Psi^\dagger\,A\,\alpha^\lambda\,\alpha^{[\mu\nu]}\,\Psi.\,\sqrt{g},$$

so

$$R_\nu^{\lambda\mu} = Re - \frac{hc}{4\,i}\,\Psi^\dagger\,A\,\alpha^\lambda\,\alpha^\mu\,\alpha_\nu\,\Psi.\,\sqrt{g}.$$

(Here we have made use of the fact that $Re\,\dfrac{1}{i}\,\Psi^\dagger\,A\,\alpha^\mu\,\Psi = 0$).

For the tensor $T_{\mu\nu}$ we then find, using the field equations:

$$T_{\mu\nu} = Re\,\frac{hc}{2\,i}\,\{\Psi^\dagger A\,(\alpha_\mu\,\Psi_{||\nu} + \alpha_\nu\,\Psi_{||\mu}) - \eta\,\Psi^\dagger A\,\alpha_0\,(\alpha_\mu X_\nu + \alpha_\nu X_\mu)\,\Psi\}, \quad (94)$$

where

$$\alpha_0 = \alpha^\mu\,X_\mu. \quad (95)$$

Finally we must establish the connection between 5-undors and 4-undors to show the equivalence of (92) with the DIRAC-equation in 4-space. To this purpose we introduce 4 matrices a_i by

$$a_i = \gamma^\nu_{.i}\,\alpha_\nu \quad (96)$$

which satisfy (cf. (21a) and (31))

$$a_i\,a_k + a_k\,a_i = 2\,g_{ik}, \quad (96a)$$

$$a_i\,\alpha_0 + \alpha_0\,a_i = 0. \quad (96b)$$

From (33) we get

$$A\,a_k = (A\,a_k)^\dagger,\ (A\,\alpha_0) = (A\,\alpha_0)^\dagger. \quad (96c)$$

The connection between 5-undors Ψ and 4-undors ψ is assumed to be given by

$$\Psi = \psi \,.\, F^{l}; \quad \Psi^{\dagger} = \psi^{\dagger} \,.\, F^{-l} \tag{97}$$

where ψ is a homogeneous function of X^{μ} of degree zero, F the homogeneous function of degree one occurring in (29) and l is given by (89*a*).

We also must find the connection between the Λ_{μ} and the analogous 4-space quantities occurring in the ordinary covariant derivative of α_k:

$$\alpha_{k\,||\,l} = \alpha_{k\,|\,l} - \overset{4}{\Gamma}{}^{i}_{kl}\, \alpha_i + \Lambda_l\, \alpha_k - \alpha_k\, \Lambda_l = 0.$$

It is found *) that

$$\Lambda_{\mu} = \gamma_{\mu}^{\cdot\, i}\, \Lambda_i + \triangle_{\mu}, \tag{98}$$

where

$$\triangle_{\mu} = -\tfrac{1}{4}\, X_{\mu l}\, \alpha_0\, \alpha^{l} - \tfrac{1}{16}\, X_{\mu}\, X_{kl}\, \alpha^{[kl]}. \tag{99}$$

With the help of (29), (90), (96), (97) and (98) we now can derive the DIRAC-equation in 4-space:

$$\alpha^{k}\,(\psi_{||k} - l f_k\, \psi) + \eta\, \psi + \tfrac{1}{16}\, X_{kl}\, \alpha_0\, \alpha^{[kl]}\, \psi = 0,$$

or

$$\alpha^{k}\, \psi_{|\lfloor k} + \frac{imc}{h}\, \psi + \frac{\sqrt{2\varkappa}}{16c}\, F_{kl}\, \alpha_0\, \alpha^{[kl]}\, \psi = 0. \tag{100}$$

Here

$$\psi_{|\lfloor k} = \psi_{||k} - \frac{ie}{hc}\, \varphi_k\, \psi \tag{101}$$

is the gauge-invariant covariant derivative of ψ. The term proportional to $\sqrt{\varkappa}$ is so small that its physical consequences (magnetic moment for uncharged particles with spin) are negligible.

From (95) we can derive the 4-dimensional energy momentum tensor and the charge current density, using the prescriptions (20*a*) and (82). This calculation goes in the same way as in PAULI's case and we will give here only the results:

---

*) See PAULI [6]), p. 361.

$$T_{ik} = Re\,\frac{h\,c}{2\,i}\,\psi^{\dagger} A\,(a_k\,\psi_{||i} + a_i\,\psi_{||k}) +$$

$$+ \frac{\sqrt{2\varkappa}}{16c} \cdot \frac{h\,c}{i}\,\psi^{\dagger} A\,(a_{[kl]}\,a_0\,F_i{}^l + a_{[il]}\,a_0\,F_k{}^l)\,\psi, \quad (102)$$

$$s^i = e\,\psi^{\dagger} A\,a^i\,\psi - \frac{\sqrt{2\varkappa}}{8c} \cdot \frac{h\,c}{i} \cdot \frac{1}{\sqrt{g}}(\sqrt{g}\,\psi^{\dagger} A\,a_0\,a^{[ik]}\,\psi)_{|k}. \quad (103)$$

(102) and (103) differ from the corresponding expressions found with the usual methods by small terms. As already pointed out, it is impossible to decide empirically whether this has to be regarded as a defect of projective relativity theory in its present form. Finally we remark that one can easily verify that s^i satisfies (84), noting that the divergence of the second term in (103) vanishes on account of $a^{[ik]} = -a^{[ki]}$ and using the field equations.

REFERENCES.

1) TH. KALUZA, S-B. Akad. Wiss. Berlin **1921**, 966.
2) O. KLEIN, Z. Phys. **37**, 895, 1926; **46**, 188, 1927.
3) A. EINSTEIN and W. MAYER, S-B. Akad. Wiss. Berlin, **1931**, 541; **1932**, 130.
4) O. VEBLEN and B. HOFFMANN, Phys. Rev. **36**,810, 1931.
5) J. A. SCHOUTEN, Ann. Inst. H. Poincaré, **5**, 49, 1935 (with references to previous work by Schouten and van Dantzig).
6) W. PAULI, Ann. Physik, **18**, 305, 337, 1933.
7) F. J. BELINFANTE, Physica, **7**, 449, 1940.
8) L. ROSENFELD, Mém. Acad. roy. Belgique, **18**, fasc. 6, 1940.
9) D. HILBERT, Gött. Nachr. **1915**, 395.
10) H. TETRODE, Z. Phys. **49**, 858, 1928.
11) E. SCHRÖDINGER, S-B. Akad. Wiss. Berlin, **1932**, 105.
12) H. WEYL, Z. Phys. **56**, 330, 1929.
13) F. J. BELINFANTE, Physica, **7**, 305, 1940.

CHAPTER II.

MESON FIELDS IN 5-DIMENSIONAL PROJECTIVE SPACE.

Summary.

The theory of projective relativity is applied to meson fields; it is shown how to incorporate the MøLLER–ROSENFELD theory of nuclear forces in this scheme. The two main features of such a treatment are: 1. a reduction of the number of universal constants in the mentioned theory; 2. the automatical introduction of the interaction between mesons and the electromagnetic field. After it has been shown how to deal with the electron-neutrino field within this formalism, an expression for the energy momentum 5-tensor is derived from which one can obtain the Hamiltonian and the charge current density of the system. The commutation rules for meson field variables are also brought in a more compact form. The Hamiltonian is then transformed by separating off the longitudinal electromagnetic field and the static meson field successively and the transformation of the current and the density of electric charge is discussed in detail. Finally, expressions are given for the electric dipole and quadrupole moment and for the magnetic dipole moment of a nuclear system.

§ 1. *Introduction.* The idea to describe nuclear forces by a charged field, corresponding with particles (mesons) of integral spin and mass intermediate between those of electron and nucleon *) was first put forward by YUKAWA. The scalar field (meson spin zero) originally introduced for this purpose [1]) does not give the right picture of these forces, but the introduction of mesons of other type may help to overcome this difficulty. KEMMER [2]) has namely shown that, assuming the spin of the mesons to be not greater than one, there are four kinds of possible meson fields characterized by the transformation properties of the field variables. One may then use either a vector field (spin 1) or a pseudoscalar field (spin 0) or some suitable combination of them. Further, the best way to account for the practical equality of proton-proton and proton-

*) BELINFANTE has suggested to call the heavy particle of which proton and neutron are different states a "nuclon". However, if we keep to the custom of using the ending "—on" for names of elementary particles, the correct form of the word is, of course, "nucleon".

neutron forces seems to be to introduce neutral mesons besides the charged ones in a symmetrical way *), as proposed by KEMMER [3]).

Adopting this last assumption, MøLLER and ROSENFELD [5]) (this paper will in the following be quoted as M.R.) have especially advocated a particular combination of a vector and a pseudoscalar field which allows to eliminate a term of highly singular character (dipole interaction potential) from the expression for the static nuclear interaction. The strength of the coupling between nucleons and these fields is described by four constants: two, characteristic for the "vector-interaction", ($g_1^{M.R.}$, $g_2^{M.R.}$) and two for the "pseudoscalar interaction", ($f_1^{M.R.}$, $f_2^{M.R.}$). Apart from the condition $|g_2^{M.R.}|^2 = |f_2^{M.R.}|^2$ that is necessary to eliminate the dipole potential, they are completely independent. In view of the possibility to obtain stringent tests for this theory, arguments which would enable us to reduce this number of constants would be very welcome.

Recently, MøLLER [6]) **) has pointed out that such a reduction follows from the requirement that the M.R.-theory be invariant with respect to a wider group of transformations than the Lorentz-group, namely that of the rotations in a five-dimensional space. Moreover, it is then possible to bring the field-equations in a more compact form. MøLLER chooses for this space on whose properties the theory now essentially depends the five-dimensional DE SITTER space. This, however, seems not to give rise to any significant physical consequences. On the other hand, the treatment of this problem from the projective point of view has the advantage that, besides the reduction of the constants, the interaction of mesons with the electromagnetic field is automatically introduced. This will be studied in the present chapter.

Following M.R., we describe the mesons by three real fields. The quantities referring to each field are distinguished by a bold-printed index, e.g.

$$F_{\mathbf{1}},\ F_{\mathbf{2}},\ F_{\mathbf{3}},$$

where $F_{\mathbf{1}}$ and $F_{\mathbf{2}}$ represent the charged and $F_{\mathbf{3}}$ the neutral mesons.

---

*) Another possibility has been proposed by BETHE [4]).

**) I am much indebted to dr. MøLLER for the communication of his results before publication.

Three such quantities are then written as $\mathbf{F}$, $\mathbf{F}$ being, (with regard to the index $\mathbf{i}$), a vector in "isotopic spin space". In the same way the nuclear source densities $\mathbf{S}$ which determine the real fields in question are treated:

$$\mathbf{S} = (S_1, S_2, S_3).$$

$\mathbf{S}$ can be brought into the form

$$\mathbf{S} = \boldsymbol{\tau} S,$$

where S is the same for the three fields, and τ the isotopic spin vector, (the eigenvalue $+1$ (-1) of τ_3 denoting neutron (proton) states). Further we need for the following similar symbolical forms for electromagnetic field quantities. If φ_i is the electromagnetic vector potential we put

$$\boldsymbol{\varphi}_i = (0, 0, \varphi_i). \tag{1}$$

As in I we will denote the derivative of a tensor $\mathbf{F}$ by $\mathbf{F}_{|\mu}$ (in 5-space) or $\mathbf{F}_{|i}$ (in 4-space), the covariant derivative by $\mathbf{F}_{||\mu}$ $(\mathbf{F}_{||i})$. The covariant gauge derivative *) is indicated by $\mathbf{F}_{|\lfloor\mu}$ $(\mathbf{F}_{|\lfloor i})$ and we have:

$$\mathbf{F}_{|\lfloor i} = \mathbf{F}_{||i} - \frac{e}{hc} \boldsymbol{\varphi}_i \wedge \mathbf{F}, \tag{2a}$$

$$\mathbf{F}_{|\lfloor\mu} = \mathbf{F}_{||\mu} - \frac{e}{hc} \boldsymbol{\varphi}_\mu \wedge \mathbf{F} \; ; \; \boldsymbol{\varphi}_\mu = \gamma_{\mu}^{\cdot k} \boldsymbol{\varphi}_k. \tag{2b}$$

The symbols $\wedge$ and $\boldsymbol{\wedge}$ indicate a vector product in ordinary and symbolical space respectively. Thus

$$(\boldsymbol{\varphi}_i \wedge \mathbf{F})_1 = -\varphi_i F_2, \quad (\boldsymbol{\varphi}_i \wedge \mathbf{F})_2 = \varphi_i F_1, \quad (\boldsymbol{\varphi}_i \wedge \mathbf{F})_3 = 0.$$

§ 2. *The meson field in the absence of other material fields.* The mesons are described by a 5-vector $\mathbf{U}_\mu$ and an antisymmetrical 5-tensor $\mathbf{F}_{\mu\nu}$ defined by **)

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \boldsymbol{\Phi}_{\mu\nu} - \lambda \mathbf{X}_{[\mu} \wedge \mathbf{U}_{\nu]}, \\ \boldsymbol{\Phi}_{\mu\nu} &= \mathbf{U}_{\nu||\mu} - \mathbf{U}_{\mu||\nu} = \mathbf{U}_{\nu|\mu} - \mathbf{U}_{\mu|\nu}. \end{aligned} \tag{3}$$

---

*) We prefer this name to "covariant gauge-invariant derivative" as the latter might suggest the invariance of meson theory with respect to the group of gauge transformations which in fact is not the case.

**) $a_{[\mu} b_{\nu]} = a_\mu b_\nu - a_\nu b_\mu$.

The components of $\mathbf{X}_\mu$ in isotopic spin space are $(0, 0, X_\mu)$; λ is a for the present undetermined constant. For the Lagrangian we put *)

$$L = -\tfrac{1}{4}\,\mathbf{F}_{\mu\nu}\,\mathbf{F}^{\mu\nu} - \frac{\mu^2}{2}\,\mathbf{U}_\mu\,\mathbf{U}^\mu, \tag{4}$$

$\mu = m_0 c/h$, m_0: meson rest mass,

so the field equations are

$$\mathbf{F}^{\mu\nu}{}_{||\nu} = -\mu^2\,\mathbf{U}^\mu - \lambda\,\mathbf{F}^{\mu\nu} \wedge \mathbf{X}_\nu. \tag{5}$$

L is invariant with respect to rotations in isotopic spin space, and we will now perform a rotation around the 3-axis of that space (phase transformation). If $\mathbf{Q}_{(\alpha)}$ is a field variable and $\mathbf{q}_{(\alpha)}$ the transformed quantity, such a transformation is given by

$$Q_{(\alpha),\mathrm{i}} = \Theta_\mathrm{i}^\mathrm{k}\, q_{(\alpha),\mathrm{k}}, \tag{6}$$

$$\Theta_\mathrm{i}^\mathrm{k} = \begin{pmatrix} \cos x, & -\sin x, & 0 \\ \sin x, & \cos x, & 0 \\ 0, & 0, & 1 \end{pmatrix}; \tag{6a}$$

we put

$$x = \lambda \log F, \tag{6b}$$

where F is an arbitrary homogeneous function of the first degree in X^μ.

The product of two field variables transforms as follows

$$\mathbf{Q}_{(\alpha)}\,\mathbf{Q}_{(\beta)} = \mathbf{q}_{(\alpha)}\,\mathbf{q}_{(\beta)},$$

so (4) becomes

$$L = -\tfrac{1}{4}\,\mathbf{f}_{\mu\nu}\,\mathbf{f}^{\mu\nu} - \frac{\mu^2}{2}\,\mathbf{u}_\mu\,\mathbf{u}^\mu. \tag{7}$$

As a consequence of (6) also (3) and (5) are affected. Considering (3) we have

$$\Phi_{\mathrm{i},\mu\nu} = \Theta_\mathrm{i}^\mathrm{k}\,\phi_{\mathrm{k},\mu\nu} = \Theta_\mathrm{i}^\mathrm{k}\,(u_{\mathrm{k},\nu|\mu} - u_{\mathrm{k},\mu|\nu}) + (\Theta_{\mathrm{i}}^{\mathrm{k}}{}_{|\mu}\, u_{\mathrm{k},\nu} - \Theta_{\mathrm{i}}^{\mathrm{k}}{}_{|\nu}\, u_{\mathrm{k},\mu}).$$

---

*) $\mathbf{AB} = \sum_\mathrm{i} A_\mathrm{i}\, B_\mathrm{i}$. See also the last footnote on p. 22.

Now by (6*b*)

$$x_{|\mu} = \frac{\lambda}{F} F_{|\mu},$$

thus if we identify F with the corresponding function in I, (29):

$$x_{|\mu} = \lambda (X_\mu - \bar{f}_\mu),$$

we get

$$\Phi_{\mu\nu} = \mathbf{u}_{\nu|\mu} - \mathbf{u}_{\mu|\nu} + \lambda \mathbf{X}_{[\mu} \wedge \mathbf{u}_{\nu]} - \lambda \bar{\mathbf{f}}_{[\mu} \wedge \mathbf{u}_{\nu]}.$$

Consequently, putting

$$\lambda = \frac{e}{hc} \cdot \frac{c}{\sqrt{2\varkappa}}, \text{ ($\varkappa$ the gravitational constant),} \tag{8}$$

and using (2*b*) and I (28), formula (3) takes the form

$$\mathbf{f}_{\mu\nu} = \mathbf{u}_{\nu||\mu} - \mathbf{u}_{\mu||\nu}. \tag{9}$$

In the same way (5) must be treated. The result is

$$\mathbf{f}^{\mu\nu}{}_{||\nu} = -\mu^2 \mathbf{u}^\mu. \tag{10}$$

From each equation (9) and (10) two four-dimensional equations can be derived. If we introduce the 4-tensor $\mathbf{f}_{ik}$ and the 4-vector $\mathbf{g}_i$ (cf. I, (20*a*)):

$$\mathbf{f}_{ik} = \gamma^\mu_{.i} \gamma^\nu_{.k} \mathbf{f}_{\mu\nu} = -\mathbf{f}_{ki}, \tag{11}$$

$$\mathbf{g}_i = X^\mu \gamma^\nu_{.i} \mathbf{f}_{\mu\nu}, \tag{12}$$

and the 4-vector $\mathbf{u}_k$ and the scalar $\mathbf{u}$ (cf. I (19)) by:

$$\mathbf{u}_k = \gamma^\mu_{.k} \mathbf{u}_\mu, \tag{13}$$

$$\mathbf{u} = X^\mu \mathbf{u}_\mu, \tag{14}$$

we have

$$\mathbf{f}_{\mu\nu} = \gamma_\mu^{.i} \gamma_\nu^{.k} \mathbf{f}_{ik} + X_{[\mu} \gamma_{\nu]}^{.i} \mathbf{g}_i, \tag{15a}$$

$$\mathbf{u}_\mu = \gamma_\mu^{.k} \mathbf{u}_k + \mathbf{u} X_\mu. \tag{16a}$$

In order to obtain four-dimensional equations from (9) we first contract this equation with $\gamma^\mu_{.i} \gamma^\nu_{.k}$. With the help of I (23) we obtain

$$\mathbf{f}_{ik} = \varphi_{ik} + \mathbf{u} X_{ik}; \qquad \varphi_{ik} = \mathbf{u}_{k||i} - \mathbf{u}_{i||k}. \tag{17a}$$

Next we contract with $X^\mu \gamma^\nu_{.i}$ and use (see I (13), (13a))

$$X^\mu \mathbf{u}_{\nu||\mu} = -\tfrac{1}{2} X_\nu^{\cdot\mu} \mathbf{u}_\mu, \qquad X^\mu{}_{||\nu} = \tfrac{1}{2} X_\nu^{\cdot\mu}.$$

Therefore, $(X^\mu \bar{f}_\mu = 0)$:

$$X^\mu \gamma^\nu_{.i} \mathbf{u}_{\nu||\lfloor\mu} = X^\mu \gamma^\nu_{.i} \mathbf{u}_{\nu||\mu} = -\tfrac{1}{2} X_i^{\cdot k} \mathbf{u}_k,$$

$$X^\mu \gamma^\nu_{.i} \mathbf{u}_{\mu||\lfloor\nu} = \mathbf{u}_{||\lfloor i} - \tfrac{1}{2} X_i^{\cdot k} \mathbf{u}_k.$$

So

$$\mathbf{g}_i = -\mathbf{u}_{||\lfloor i}. \tag{18a}$$

In the same way we proceed with (10) and get

$$\mathbf{f}^{ik}{}_{||\lfloor k} = -\mu^2 \mathbf{u}^i, \tag{19a}$$

$$\mathbf{g}^k{}_{||\lfloor k} + \tfrac{1}{2} X^{ik} \mathbf{f}_{ik} = -\mu^2 \mathbf{u}. \tag{20a}$$

By means of the connection (16*a*) we thus have obtained from (3) and (4) a mixture of a vector and a scalar meson field. One can, however, modify (15*a*) and (16*a*) in a covariant way, such that, in the case of special relativity *), (3) and (4) give rise to a set of vector and *pseudo*scalar field equations.

Consider the quantity

$$\varepsilon = \frac{\gamma_{[\lambda_1}^{\cdot[i_1} \gamma_{\lambda_2}^{\cdot i_2} \gamma_{\lambda_3}^{\cdot i_3} \gamma_{\lambda_4}^{\cdot i_4]} X_{\lambda_5]}}{\left| \gamma_{[\lambda_1}^{\cdot[i_1} \gamma_{\lambda_2}^{\cdot i_2} \gamma_{\lambda_3}^{\cdot i_3} \gamma_{\lambda_4}^{\cdot i_4]} X_{\lambda_5]} \right|}, \tag{21}$$

where the brackets denote antisymmetrization with respect to the embraced indices. The tensor in the numerator **) has only one "Kennzahl": η. Obviously η is a scalar with regard to the group of 5-dimensional rotations that have the Det. $+1$. Furthermore, it is a pseudoscalar with respect to the full Lorentz-group as follows from well known considerations. Therefore the constant ε has the same properties, while moreover:

$$\varepsilon^2 = 1. \tag{21a}$$

---

*) We have chosen this particular formulation in order to show clearly the connection with special relativity which only is of interest for our present purpose. It is clear that a quite general formulation is possible; to this point we hope to return later.

**) This tensor occurs in a paper by SCHOUTEN, *loc. cit.* [7]), p. 60 equ. (16). I thank dr. PODOLANSKI for drawing my attention to this formula.

Putting

$$\mathbf{w} = \varepsilon \mathbf{u}, \quad \mathbf{h}_i = \varepsilon \mathbf{g}_i,$$

we may write instead of (15*a*) and (16*a*)

$$\mathbf{f}_{\mu\nu} = \gamma_{\mu}^{;i} \gamma_{\nu}^{;k} \mathbf{f}_{ik} + \varepsilon X_{[\mu} \gamma_{\nu]}^{;i} \mathbf{h}_i, \tag{15b}$$

$$\mathbf{u}_\mu = \gamma_{\mu}^{;k} \mathbf{u}_k + \varepsilon \mathbf{w} X_\mu, \tag{16b}$$

which in the case of special relativity denotes a decomposition of a 5-tensor in a 4-tensor and a pseudo 4-vector and of a 5-vector in a 4-vector and a 4-pseudoscalar. Starting from (15*b*), (16*b*) we now obtain from (3) and (4):

$$\mathbf{f}_{ik} = \Phi_{ik} + \varepsilon \mathbf{w} X_{ik}, \tag{17b}$$

$$\mathbf{h}_i = -\mathbf{w}_{||i}, \tag{18b}$$

$$\mathbf{f}^{ik}{}_{||k} = -\mu^2 \mathbf{u}^i, \tag{19b}$$

$$\mathbf{h}^k{}_{||k} + \tfrac{1}{2} \varepsilon X_{ik} \mathbf{f}^{ik} = -\mu^2 \mathbf{w}. \tag{20b}$$

The equations (17)—(20) differ from those derived by the usual methods (compare e.g. BHABHA [8]) by terms proportional to some power of the gravitational constant (on account of I (27)). These terms define an interaction between vector and scalar (or pseudo-scalar) mesons, but only through the intermediary of the electromagnetic field. We need not bother about these somewhat peculiar terms, however, as they are too small to have any effect on practical calculations.

Finally we will write down the Lagrangian (7) expressed in 4-dimensional quantities in case we start from (15*b*) and (16*b*); using (21*a*):

$$L = -\tfrac{1}{4} \mathbf{f}_{ik} \mathbf{f}^{ik} - \frac{\mu^2}{2} \mathbf{u}_k \mathbf{u}^k - \tfrac{1}{2} \mathbf{h}_k \mathbf{h}^k - \frac{\mu^2}{2} \mathbf{w}^2. \tag{7a}$$

The first term on the right contains the small interactions mentioned above.

§ 3. *Interaction with nucleons.* We introduce a 5-tensor $\mathbf{S}_{\mu\nu}$ and a 5-vector $\mathbf{M}_\mu$:

$$\mathbf{S}_{\mu\nu} = \frac{g_2}{2\mu} \Psi^\dagger \boldsymbol{\tau} A \, a_{[\mu\nu]} \Psi, \tag{22}$$

$$\mathbf{M}_\mu = g_1 \Psi^\dagger \boldsymbol{\tau} A \, a_\mu \Psi. \tag{23}$$

The hermitizing matrix A (I, (33)) makes both quantities real; Ψ is the nucleon 5-undor, having 8 components.

In this case we assume the Lagrangian to be

$$L_{\text{tot}} = L + L_n,$$

$$L = -\tfrac{1}{4}\,\mathbf{F}_{\mu\nu}\,\mathbf{F}^{\mu\nu} - \frac{\mu^2}{2}\,\mathbf{U}_\mu\,\mathbf{U}^\mu + \mathbf{U}_\mu\,\mathbf{M}^\mu \tag{24}$$

and L_n is the Lagrangian for free nucleons:

$$L_n = Re \quad i h c\,(\Psi^\dagger A\,\alpha^\mu\,\Psi_{||\mu} + \eta\,\Psi^\dagger A\,\Psi) \tag{25}$$

with

$$\eta = \frac{i}{hc}\left(\frac{1+\tau_3}{2}\,m_N\,c^2 + \frac{1-\tau_3}{2}\,m_p\,c^2\right),$$

(cf. I, (93); m_N and m_P denote the masses of neutron and proton respectively). $\mathbf{F}_{\mu\nu}$ is defined by:

$$\mathbf{F}_{\mu\nu} = \boldsymbol{\Phi}_{\mu\nu} - \lambda\,\mathbf{X}_{[\mu} \wedge \mathbf{U}_{\nu]} + \mathbf{S}_{\mu\nu}, \tag{26}$$

so the field equations are

$$\mathbf{F}^{\mu\nu}{}_{||\nu} = -\mu^2\,\mathbf{U}^\mu + \mathbf{M}^\mu - \lambda\,\mathbf{F}^{\mu\nu} \wedge \mathbf{X}_\nu. \tag{27}$$

In the same way as before we transform the Lagrangian by (6). Consequently, in the terms of L arising from the presence of nucleons we have to replace τ_i by τ_i'. This rotation in isotopic spin space can, however, be compensated by a change of representation in this space. Having performed this we can continue to operate with τ_i. Thus (26) and (27) become:

$$\mathbf{f}_{\mu\nu} = \mathbf{u}_{\nu\,|\lfloor\mu} - \mathbf{u}_{\mu\,|\lfloor\nu} + \mathbf{S}_{\mu\nu}, \tag{26$'$}$$

$$\mathbf{f}^{\mu\nu}{}_{|\lfloor\nu} = -\mu^2\,\mathbf{u}^\mu + \mathbf{M}^\mu. \tag{27$'$}$$

To obtain 4-dimensional equations from (26′), (27′) we need the 4-space quantities that can be derived from α_μ. In analogy with (16*a*) and (16*b*) there are two possible decompositions, giving rise to a mixture of a vector and a scalar and of a vector and a pseudo-scalar meson field respectively:

1° $$\alpha_\mu = \gamma_\mu^{\cdot k}\,\alpha_k + \alpha_0\,X_\mu, \tag{28a}$$

where α_k, α_0 satisfy I, (95) and (96),

2° $$\alpha_\mu = \gamma_\mu^{\cdot k}\,\alpha_k + \varepsilon\,\beta_0\,X_\mu, \qquad \beta_0 = \varepsilon\,\alpha_0. \tag{28b}$$

β_0 is, according to its definition, a 4-pseudoscalar; from I, (96) it follows that

$$a_i\,\beta_0 + \beta_0\,a_i = 0, \qquad \beta_0^2 = 1, \qquad A\,\beta_0 = (A\,\beta_0)^\dagger. \tag{29}$$

In fact, in the case of special relativity, $(g_{11} = g_{22} = g_{33} = -g_{44} = 1)$, a realisation of a_i, a_0, β_0 and A in accordance with I (96) and (29) is given by

$$\left.\begin{aligned} a_i &= \varrho_3\,\sigma_i \\ a_4 &= -i\,\varrho_2 \\ a_0 &= -\varrho_1 \\ A &= -i\,\varrho_2 \end{aligned}\right\}\ \text{for (28}a\text{)}, \qquad \left.\begin{aligned} a_i &= \varrho_2\,\sigma_i \\ a_4 &= i\,\varrho_3 \\ \beta_0 &= -\varrho_1 \\ A &= i\,\varrho_3 \end{aligned}\right\}\ \text{for (28}b\text{)}; \tag{30}$$

both columns refer to the *same* representation of ϱ_i, σ_i. Consequently in either case $\psi^\dagger A a_i \psi$ is a vector, while $\psi^\dagger A a_0 \psi$ is a scalar and $\psi^\dagger A \beta_0 \psi$ is a pseudoscalar *). In the following only the field equations corresponding with (28b) will be considered **). These are

$$\mathbf{f}_{ik} = \varphi_{ik} + \mathbf{S}_{ik} + \varepsilon\,\mathbf{w}\,X_{ik}, \tag{31}$$

$$\mathbf{S}_{ik} = \frac{g_2}{2\mu}\,\psi^\dagger\,\tau\,A\,a_{[ik]}\,\psi, \tag{31a}$$

$$\mathbf{h}_i = -\mathbf{w}_{|\lfloor i} + \mathbf{S}_i^{(0)}, \tag{32}$$

$$\mathbf{S}_i^{(0)} = \frac{g_2}{\mu}\,\psi^\dagger\,\tau\,A\,\beta_0\,a_i\,\psi, \tag{32a}$$

$$\mathbf{f}^{ik}{}_{|\lfloor k} = -\mu^2\,\mathbf{u}^i + \mathbf{M}^i, \tag{33}$$

$$\mathbf{M}^i = g_1\,\psi^\dagger\,\tau\,A\,a_i\,\psi, \tag{33a}$$

$$\mathbf{h}^k{}_{|\lfloor k} + \tfrac{1}{2}\,\varepsilon\,X^{kl}\,\mathbf{f}_{kl} = -\mu^2\,\mathbf{w} + \mathbf{M}^{(0)}, \tag{34}$$

$$\mathbf{M}^{(0)} = g_1\,\psi^\dagger\,\tau\,A\,\beta_0\,\psi. \tag{34a}$$

---

*) The S-transformation corresponding with spatial reflections is in both cases: $S = \varrho_3$.

**) To the general problem of the different kinds of meson fields that are admitted by the projective formalism and by MøLLER's non-projective theory we hope to return later.

In order to compare this set of equations with the M.R. formalism we adopt a similar notation as used there for the source densities:

$$\mathrm{S}_{ik} = \vec{\mathrm{T}}, \vec{\mathrm{S}}, \qquad \mathrm{S}_i^{(0)} = \vec{\mathrm{P}}, -\mathrm{Q},$$

$$\mathbf{M}_i = \vec{\mathbf{M}}, -\mathbf{N}, \quad \mathbf{M}^{(0)} = \mathbf{R}.$$

The expressions (31*a*)—(34*a*), (in configuration space of the nucleons), become identical with those of M.R. if we put

$$g_1 = f_1^{\mathrm{M.R.}} = g_2^{\mathrm{M.R.}} \quad ; \quad g_2 = f_2^{\mathrm{M.R.}} = -g_2^{\mathrm{M.R.}}.$$

Thus in order to describe the forces between nucleons we only need *two* constants which are related to those in M.R. in the same way as the two constants of MøLLER's non-projective formalism.

§ 4. *Interaction with the electron-neutrino field.* It is also possible to incorporate in the present scheme the interaction of meson fields with a system of light particles (electrons and neutrino's) and we will shortly indicate the way of treatment.

Following ROZENTAL *) [9]) we describe the light particles by three real fields with the help of an "isotopic spin vector" $\breve{\tau}$, $\breve{\tau}_3 = +1\ (-1)$ referring to the electron (neutrino) state of the particle. Then, similarly to (22) and (23) we can form a 5-tensor $\breve{\mathbf{S}}_{\mu\nu}$ and a 5-vector $\breve{\mathbf{M}}_\mu$

$$\breve{\mathbf{S}}_{\mu\nu} = \frac{\breve{g}_2}{2\mu} \breve{\Psi}^\dagger \breve{\tau} \breve{A} \breve{a}_{[\mu\nu]} \breve{\Psi},$$

$$\breve{\mathbf{M}}_\mu = \breve{g}_1 \breve{\Psi}^\dagger \breve{\tau} \breve{A} \breve{a}_\mu \breve{\Psi},$$

$\breve{\Psi}$ being the electron-neutrino 5-undor. Instead of (26) we now define $\mathbf{F}_{\mu\nu}$ by

$$\mathbf{F}_{\mu\nu} = \Phi_{\mu\nu} - \lambda \mathbf{X}_{[\mu} \wedge \mathbf{U}_{\nu]} + \mathbf{S}_{\mu\nu} + \breve{\mathbf{S}}_{\mu\nu}$$

---

*) I should like to thank dr. ROZENTAL for the kind communication of his manuscript.

and also add a term $\mathbf{U}_\mu \check{\mathbf{M}}^\mu$ and a term referring to the free electron-neutrino field to the Lagrangian (24). As we are free to introduce in the Lagrangian scalars that do not influence the field equations it is further possible to add a term to L of the form

$$\alpha \mathbf{S}_{\mu\nu} \check{\mathbf{S}}^{\mu\nu} + \beta \mathbf{M}_\mu \check{\mathbf{M}}^\mu,$$

(α and β are arbitrary constants), describing a direct interaction between the nucleons and the electron-neutrino field (not involving a meson in an intermediate state).

It will be clear that in a theory which does not make use of the condition of covariance in a 5-dimensional space, one can in the most general case introduce *four* new constants instead of the two constants α and β that suffice here. ROZENTAL has shown, however, that such a diminution of the number of constants, on the basis of MØLLER's theory, does not essentially affect the general conclusions regarding the theory of β-radioactivity and meson disintegration.

§ 5. *The energy momentum tensor and the charge current density.* With the help of the prescription given in I, we will derive an expression for the energy momentum tensor, using L in its form (24) and always writing down tensors instead of tensor densities.

For $R^{\lambda\mu}_{\nu}$ (cf. I (77)) we find, using I, (46), (75) and (76)

$$R^{\lambda\mu}_{\nu} = \mathbf{F}^{\lambda\mu} \mathbf{U}_\nu.$$

The terms in L containing $\mathbf{X}^\mu$ explicitly are

$$\frac{\lambda}{2} \mathbf{X}_{[\mu} \wedge \mathbf{U}_{\nu]} (\Phi^{\mu\nu} + \mathbf{S}^{\mu\nu}) + \frac{\lambda^2}{4} \mathbf{X}_{[\mu} \wedge \mathbf{U}_{\nu]} \mathbf{X}^{[\mu} \wedge \mathbf{U}^{\nu]};$$

on account of $\mathbf{X}_\mu \mathbf{X}^\mu = X_\mu X^\mu = 1$, however, the second term is equal to

$$\frac{\lambda^2}{4} (\mathbf{U}_\nu \mathbf{U}^\nu - U_{\nu,3} U^\nu_3)$$

and in this form it does no more depend explicitly on X^μ. Therefore

$$X^\mu \frac{\partial^e L}{\partial^e X^\nu} = \mathbf{X}^\mu \frac{\partial^e L}{\partial^e \mathbf{X}^\nu} = \lambda (\mathbf{F}_{\nu\varrho} + \lambda \mathbf{X}_{[\nu} \wedge \mathbf{U}_{\varrho]}) \cdot (\mathbf{X}^\mu \wedge \mathbf{U}^\varrho)$$

with

$$\frac{\partial^e}{\partial^e \mathbf{X}^\nu} \equiv \left(0,\, 0,\, \frac{\partial^e}{\partial^e X^\nu}\right).$$

We can now immediately obtain the energy momentum 5-tensor by using I (80) and the field equations (26) and (27):

$$T^{\mu}_{.\nu} = -\mathbf{F}^{\mu\varrho}\mathbf{F}_{\nu\varrho} - \mu^2\,\mathbf{U}^\mu\,\mathbf{U}_\nu + \mathbf{F}^{\mu\varrho}\mathbf{S}_{\nu\varrho} + \mathbf{M}^\mu\,\mathbf{U}_\nu - L\,\delta^\mu_\nu -$$

$$-\lambda\{\mathbf{F}^{\mu\lambda}(\mathbf{X}_\nu \wedge \mathbf{U}_\lambda) + \mathbf{F}_{\nu\lambda}(\mathbf{X}^\mu \wedge \mathbf{U}^\lambda)\} - \lambda^2(\mathbf{X}_{[\nu} \wedge \mathbf{U}_{\varrho]})(\mathbf{X}^\mu \wedge \mathbf{U}^\varrho) + T_{(n)}{}^{\mu}_{.\nu}$$

where $T_{(n)}{}^{\mu}_{.\nu}$ is given by I, (94), (the constant η takes the value given in (25)). The application of the phase transformation (6) to $T^{\mu}_{.\nu}$ yields

$$T^{\mu}_{.\nu} = -\mathbf{f}^{\mu\varrho}\,\mathbf{f}_{\nu\varrho} - \mu^2\,\mathbf{u}^\mu\,\mathbf{u}_\nu + \mathbf{f}^{\mu\varrho}\,\mathbf{S}_{\nu\varrho} + \mathbf{M}^\mu\,\mathbf{u}_\nu - L\,\delta^\mu_\nu -$$

$$-\lambda\{\mathbf{f}^{\mu\lambda}(\mathbf{X}_\nu \wedge \mathbf{u}_\lambda) + \mathbf{f}_{\nu\lambda}(\mathbf{X}^\mu \wedge \mathbf{u}^\lambda)\} - \lambda^2(\mathbf{X}_{[\nu} \wedge \mathbf{u}_{\varrho]})(\mathbf{X}^\mu \wedge \mathbf{u}^\varrho) + T_{(n)}{}^{\mu}_{.\nu}. \quad (35)$$

The energy momentum 4-tensor T_{ik} and the charge current vector s_k can be found at once from $T_{\mu\nu}$. In fact we have, denoting the energy momentum tensor of the pure Maxwell field by $T_{(e)\,ik}$, (see also the last footnote on p. 22)

$$T_{ik} = \gamma_\mu^{.i}\,\gamma^{\nu}_{.k}\,T^{\mu}_{.\nu} + T_{(e)\,ik}\,,$$

$$s_k = \frac{\sqrt{2\varkappa}}{c}\,\gamma_\mu^{.i}\,X^\nu\,T^{\mu}_{.\nu}.$$

Using (15*b*) and (16*b*) we get

$$\gamma_\mu^{.i}\,\mathbf{f}^{\mu\varrho}\,\gamma^{\nu}_{.k}\,\mathbf{f}_{\nu\varrho} = (\gamma^{\varrho}_{.l}\,\mathbf{f}^{il} - \varepsilon\,\mathbf{h}^i\,X^\varrho)(\gamma_\varrho^{.m}\,\mathbf{f}_{km} - \varepsilon\,\mathbf{h}_k\,X_\varrho)$$

$$= \mathbf{f}^{il}\,\mathbf{f}_{kl} + \mathbf{h}^i\,\mathbf{h}_k\,.$$

The cross terms disappear on account of I, (16) and (18). Similarly

$$\gamma_\mu^{.i}\,\mathbf{f}^{\mu\varrho}\,\gamma^{\nu}_{.k}\,\mathbf{S}_{\nu\varrho} = \mathbf{f}^{il}\,\mathbf{S}_{kl} + \mathbf{h}^i\,\mathbf{S}^{(0)}_k.$$

The terms in (35) containing X_ν or X^μ explicitly disappear if we contract with $\gamma_\mu^{.i}\,\gamma^{\nu}_{.k}$. Using (13) and (14) we thus obtain

$$T^{i}_{.k} = -\mathbf{f}^{il}\mathbf{f}_{kl} + \mathbf{f}^{il}\mathbf{S}_{kl} + \mathbf{M}^i\mathbf{u}_k - \mu^2\mathbf{u}^i\mathbf{u}_k - \mathbf{h}^i\mathbf{h}_k +$$

$$+ \mathbf{h}^i\,\mathbf{S}^{(0)}_k - L\,\delta^i_k + T_{(n)}{}^{i}_{.k} + T_{(e)}{}^{i}_{.k}. \quad (36)$$

$T_{(n)ik}$ is given by I, (102). From (36) we infer that the Hamiltonian is given by *)

$$H = \int (H_{(f)\mathsf{L}} + H_{(e)})\, dv + H_{(n)\mathsf{L}},$$

$$H_{(f)\mathsf{L}} = -\mathbf{f}^{4l}\,\mathbf{f}_{4l} + \mathbf{f}^{4l}\,\mathbf{S}_{4l} + \mathbf{M}^4\,\mathbf{u}_4 - \mu^2\,\mathbf{u}^4\,\mathbf{u}_4 - \mathbf{h}^4\,\mathbf{h}_4 + \mathbf{h}^4\,\mathbf{S}_4^{(0)} - L,$$

$$H_{(n)\mathsf{L}} = \sum_i \left\{ e\,\frac{1-\tau_3^{(i)}}{2}\,\mathcal{B}^{(i)} + \varrho_1\,\vec{\sigma}^{(i)}\left(\vec{p}^{(i)} - \frac{1-\tau_3^{(i)}}{2}\,e\,\vec{\mathcal{A}}^{(i)}\right) + \frac{hc}{i}\,\eta^{(i)}\,\varrho_3^{(i)} + \omega \right\},$$

$$H_{(e)} = \tfrac{1}{2}\,(\vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2).$$

$H_{(n)\mathsf{L}}$ has been represented in configuration space, the index (i) referring to the i–th nucleon. $(\vec{\mathcal{A}}, -\mathcal{B})$ is the electromagnetic vector potential, $(\vec{\mathcal{E}}, \vec{\mathcal{H}})$ the electromagnetic field. ω is a small term which we will not write down explicitly.

In order to avoid the occurrence of singular terms of the δ-function type **), we add to the Lagrangian the scalar $\frac{1}{8}\,\mathbf{S}_{\mu\nu}\,\mathbf{S}^{\mu\nu}$. Introducing a vector notation for the dynamical meson variables:

$$\mathbf{f}_{ik} = \hat{\vec{\mathbf{F}}}, \hat{\vec{\mathbf{G}}}, \quad \mathbf{u}_k = \vec{\mathbf{U}}, -\mathbf{V},$$

$$\mathbf{h}_k = \hat{\vec{\Gamma}}, -\hat{\Phi}, \;\; \mathbf{w} = \Psi,$$

we then get

$$H_{(f)\mathsf{L}} = \tfrac{1}{2}\,\{\hat{\vec{\mathbf{F}}}^2 + \hat{\vec{\mathbf{G}}}^2 + \mu^2\,(\vec{\mathbf{U}}^2 + \mathbf{V}^2)\} - (\hat{\vec{\mathbf{F}}}\,\vec{\mathbf{T}} + \vec{\mathbf{U}}\,\vec{\mathbf{M}}) + \tfrac{1}{4}\,(\vec{\mathbf{T}}^2 - \vec{\mathbf{S}}^2) +$$
$$+ \tfrac{1}{2}\,(\hat{\vec{\Gamma}}^2 + \hat{\Phi}^2 + \mu^2\,\Psi^2) - (\mathbf{R}\,\Psi + \mathbf{Q}\,\hat{\Phi}) + \tfrac{1}{4}\,(\mathbf{Q}^2 - \mathbf{P}^2). \quad (37)$$

In the same way the field equations (31)—(34) can be treated.

The charge current density can be calculated by the same methods

*) Generally we denote the Hamiltonian of a system by H_{L} if it depends on some field variables $Q_{(\alpha)}$ and their gauge derivatives $Q_{(\alpha)\mathsf{L}i}$. The corresponding Hamiltonian depending on $Q_{(\alpha)}$ and $Q_{(\alpha)|i}$ is denoted by the same letter but without the symbol L.

**) Cf. MøLLER [6]), p. 26.

and it should be noted that it is the term $-\lambda \mathbf{f}^{\mu\lambda} (\mathbf{X}_\nu \wedge \mathbf{U}_\lambda)$ in (35) which here plays an essential part. We obtain

$$s^\mu = e\,\psi^\dagger A \,\frac{1-\tau_3}{2}\, \alpha^\mu \psi + \frac{e}{hc} (\mathbf{f}^{\nu\mu} \wedge \mathbf{u}_\nu)_3 + \xi^\mu .$$

ξ^μ is a small term which (apart from a contribution of the „Dirac-type" due to the nucleons, whose divergence vanishes, compare I, (103)) is equal to

$$\varepsilon \frac{\sqrt{2\varkappa}}{c} \gamma^\mu_{.k} \left[-\mathbf{f}^{ik}\, \mathbf{h}_k - \mu^2\, \mathbf{u}^i\, \mathbf{w} + \mathbf{f}^{ik}\, \mathbf{S}_{k\,(0)} + \mathbf{M}^i\, \mathbf{w} \right]$$

and it can be shown that $\xi^\mu{}_{||\mu} = 0$, using the field equations. On the other hand $s^\mu{}_{||\mu} = 0$ on account of the general theorem proved in I. Therefore, from now on entirely omitting all small terms, we may write

$$s^\mu = s^\mu_{\mathrm{nucl}} + s^\mu_{\mathrm{mes}}, \tag{38}$$

$$s^\mu_{\mathrm{nucl}} = e\,\psi^\dagger \frac{1-\tau_3}{2} A\, \alpha^\mu \psi, \tag{39}$$

$$s^\mu_{\mathrm{mes}} = \frac{e}{hc} (\mathbf{f}^{\nu\mu} \wedge \mathbf{u}_\nu)_3 . \tag{40}$$

With $s_k = \gamma^\mu_{.k}\, s_\mu$ and $s_k = \vec{s}, -\varrho$ this leads to the following expressions (cf. M.R. (61), the nucleon-part of $\vec{s}$ en ϱ is expressed in configuration space)

$$\vec{s} = \vec{s}_{\mathrm{nucl}} + \vec{s}_{\mathrm{mes}} = e \sum_i \frac{1-\tau_3^{(i)}}{2} \varrho_1^{(i)} \sigma^{(i)}\, \delta(\vec{x} - \vec{x}^{(i)}) + $$
$$+ \frac{e}{hc} (\vec{\hat{\mathbf{G}}} \wedge \vec{\mathbf{U}} - \vec{\hat{\mathbf{F}}} \wedge \mathbf{V} + \vec{\hat{\Gamma}} \wedge \vec{\hat{\Psi}})_3 , \tag{41}$$

$$\varrho = \varrho_{\mathrm{nucl}} + \varrho_{\mathrm{mes}} = e \sum_i \frac{1-\tau_3^{(i)}}{2}\, \delta(\vec{x} - \vec{x}^{(i)}) + \frac{e}{hc} (\vec{\mathbf{U}} \wedge \vec{\hat{\mathbf{F}}} - \Psi \wedge \hat{\Phi})_3 . \tag{42}$$

§ 6. *Quantization.* In the present formalism it is possible to bring the usual commutation rules in a more compact form which is especially suitable for practical calculations, and which clearly shows the intimate connection between vector and pseudoscalar variables. In fact, the commutation relations of both fields are all contained in

$$[f^{m}_{\mu 4},\ u'^{n}_{\nu}] = -\frac{h\,c}{i}\,\delta(\vec{x}-\vec{x}')\,\delta^{mn}\,\pi_{\mu\nu,4}^{\ \ \ .4}, \tag{43}$$

$$\pi_{\mu\nu,4}^{\ \ \ .4} = g_{\mu\nu} - \gamma_{\mu,4}\,\gamma_{\nu}^{.4}.$$

Here $f_{\mu 4} = \gamma^{\nu}_{.4} f_{\mu\nu}$; by u'_{ν} we understand that the four-dimensional quantities that can be derived from it, *viz.* u'_k and u' should be taken at the space time point $(\vec{x}'\,t)$, and similarly f_{i4} and g_4 refer to the argument $(\vec{x}, t)$. The isotopic spin indices are for convenience written at the upper side of the symbols. Contracting (43) with $\gamma^{\mu}_{.i}\,\gamma^{\nu}_{.k}$ we obtain using I, (17) and (21*a*)

$$[f^{m}_{i4}(\vec{x}, t),\ u^{n}_{k}(\vec{x}', t)] = -\frac{hc}{i}\,\delta(\vec{x}-\vec{x}')\,\delta^{mn}\,[g_{ik} - g_{i4}\,\delta^{4}_{k}];$$

this is equivalent with

$$[U^{m}_{i}(\vec{x}, t),\ F^{n}_{k}(\vec{x}', t)] = \frac{hc}{i}\,\delta(\vec{x}-\vec{x}')\,\delta^{mn}\,g_{ik} \quad (i, k = 1, 2, 3). \tag{44a}$$

Contraction with $\gamma^{\mu}_{.i}\,X^{\nu}$ gives simply the result that the vector variables commute with the pseudoscalar variables. Finally contraction with $X^{\mu}\,X^{\nu}$ gives

$$[\Phi^{m}(\vec{x}, t),\ \Psi^{n}(\vec{x}', t)] = \frac{hc}{i}\,\delta(\vec{x}-\vec{x}')\,\delta^{mn}. \tag{44b}$$

In a similar way we can deal with the quantization of the electromagnetic field. If $\mathfrak{F}^{ik}$ is the electromagnetic field (identical with F^{ik} in I, (27)) and $\mathfrak{F}^{\mu\nu}$ the corresponding 5-tensor we put

$$[\mathfrak{F}_{\mu 4},\ \varphi'_{\nu}] = -\frac{hc}{i}\,\delta(\vec{x}-\vec{x}')\,\chi_{\mu\nu,4}^{\ \ \ ;4},$$

$$\chi_{\mu\nu,4}^{\ \ \ ;4} = g_{\mu\nu} - X_{\mu}X_{\nu} - \gamma_{\mu,4}\,\gamma_{\nu}^{;4} = \gamma_{\mu,i}\,\gamma_{\nu}^{;i} - \gamma_{\mu,4}\,\gamma_{\nu}^{;4},$$

from which the single set of relations

$$[\mathfrak{H}_{i}(\vec{x}, t),\ \mathfrak{E}_{k}(\vec{x}', t)] = \frac{hc}{i}\,\delta(\vec{x}-\vec{x}')\,g_{ik} \quad (i, k = 1, 2, 3) \tag{44c}$$

follows *).

*) One might think that, on account of the connection of the electromagnetic

The commutation rule for the canonical variables of the nucleons is

$$[p^{(i)\,l},\ x^{(k)\,m}] = \frac{hc}{i}\,\delta^{(ik)}\,\delta^{lm}. \tag{45}$$

§ 7. *Transformation of the Hamiltonian. a) Separation of the longitudinal electromagnetic field.* In effecting this separation we put:

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_{||} + \vec{\mathcal{E}}_{\perp},$$
$$\vec{\mathcal{A}} = \vec{\mathcal{A}}_{||} + \vec{\mathcal{A}}_{\perp},$$
$$\mathcal{B} = \mathcal{B}_{\text{in}} + \mathcal{B}_{\text{ex}},$$

where $\vec{\mathcal{E}}_{||}$ is the longitudinal and $\vec{\mathcal{E}}_{\perp}$ is the transversal electric field (similarly for $\vec{\mathcal{A}}$); $\mathcal{B}_{\text{in}}$ is the part of the static potential which is created by the system of nucleons and meson fields, while $\mathcal{B}_{\text{ex}}$ is the contribution of other sources eventually present. We now must eliminate $\vec{\mathcal{E}}_{||}, \vec{\mathcal{A}}_{||}$ and $\mathcal{B}_{\text{in}}$ from H. This problem has been extensively dealt with by several authors, so we can confine ourselves here to giving the results.

For the first term of H we get

$$\int H_{(f)\mathrm{L}}\,dv = \int H_{(f)}\,dv - \int \vec{s}_{\text{mes}}\,\vec{\mathcal{A}}_{\perp}\,dv - W(\vec{\mathcal{A}}_{\perp}^2).$$

The last term denotes a quantity proportional to the square of the perturbation parameter $(e^2/hc)^{1/2}$ and therefore generally may be neglected. At the same time we then must replace the gauge derivatives which occur in $H_{(f)}$ by ordinary derivatives, and thus write $\vec{\mathbf{F}}, \vec{\mathbf{G}}, \vec{\Gamma}, \Phi$ instead of $\hat{\vec{\mathbf{F}}}, \hat{\vec{\mathbf{G}}}, \hat{\vec{\Gamma}}, \hat{\Phi}$, where the first group of quantities are the same as in M.R. Similarly we must proceed with $H_{(n)}$, $\vec{s}$ and ϱ.

---

and the gravitational field, the c-number character of the latter might be affected by (44c). Now from (51) follows (Cf. I (27) and (28)):

$$[X_{\mu 4}, f_{\nu}] = -\frac{hc}{i}\frac{2\varkappa}{c^2}\,\delta(\vec{x}-\vec{x}')\,\chi_{\mu\nu,4}$$

and it is easily seen that this does not give rise to any new commutation rule.

Further we have

$$H_{(n)L} = H_{(n)} - \int \vec{s}_{\text{nucl}} \vec{\mathcal{A}}_\perp \, dv$$

while the last terms of H gives

$$\int H_{(e)} \, dv = \int H_{(e)\perp} \, dv + \mathcal{G} + \int \varrho \, \mathcal{B}_{\text{ex}} \, dv, \qquad H_{(e)\perp} = \tfrac{1}{2}(\vec{\mathcal{E}}_\perp^2 + \vec{\mathcal{H}}^2).$$

Here $\mathcal{G}$ is the Coulomb energy of the system, ϱ is the total charge density as given by (42). Infinite electrostatic self energy terms have been suppressed. Consequently the total result is (in the following we write $\vec{\mathcal{E}}$, $\vec{\mathcal{A}}$, $\mathcal{B}$, $H_{(e)}$ instead of $\vec{\mathcal{E}}_\perp$, $\vec{\mathcal{A}}_\perp$, $\mathcal{B}_{\text{ex}}$, $H_{(e)\perp}$ respectively)

$$H = \int H_{(f)} \, dv + H_{(n)} + \int H_{(e)} \, dv + \mathcal{G} - \int \vec{s} \vec{\mathcal{A}} \, dv + \int \varrho \, \mathcal{B} \, dv \qquad (46)$$

with

$$H_{(f)} = \tfrac{1}{2} \{ \vec{\mathbf{F}}^2 + \vec{\mathbf{G}}^2 + \mu^2 (\vec{\mathbf{U}}^2 + \mathbf{V}^2) \} - (\vec{\mathbf{F}} \vec{\mathbf{T}} + \vec{\mathbf{U}} \vec{\mathbf{M}}) +$$
$$+ \tfrac{1}{2} (\vec{\Gamma}^2 + \Phi^2 + \mu^2 \Psi^2) - (\mathbf{R} \Psi + \mathbf{Q} \Phi) + \tfrac{1}{4} (\vec{\mathbf{T}}^2 - \vec{\mathbf{S}}^2) + \tfrac{1}{4} (\mathbf{Q}^2 - \vec{\mathbf{P}}^2). \qquad (47)$$

$$H_{(n)} = \sum_i \left\{ \varrho_1^{(i)} \vec{\sigma}^{(i)} \vec{p}^{(i)} + \varrho_3^{(i)} \left(\frac{1 + \tau_3^{(i)}}{2} m_N c^2 + \frac{1 - \tau_3^{(i)}}{2} m_P c^2 \right) \right\}. \qquad (48)$$

If we suppose that all wave lengths occurring in the Fourier-development of the electromagnetic field are large compared to the dimensions of the nuclear system, it may be shown that [10])

$$\int \varrho \, \mathcal{B} \, dv = \varepsilon \, \mathcal{B}_0 + \vec{\mathcal{P}} \, grad_0 \, \mathcal{B} + (\mathcal{Q} \, grad_0) \, grad_0 \, \mathcal{B}_0, \qquad (49)$$

$$\int \vec{s} \vec{\mathcal{A}} \, dv = \vec{\mathcal{A}}_0 \dot{\vec{\mathcal{P}}} + \vec{\mathcal{M}} \vec{\mathcal{H}}_0 + (\dot{\mathcal{Q}} \, grad_0) \vec{\mathcal{A}}_0, \qquad (50)$$

where the index 0 indicates that we have to take the value of the quantity at a fixed point of the nuclear system, (its centre of gravity, say). Further we have introduced the following quantities referring to the nuclear system:

its total charge: $\varepsilon = \int \varrho \, dv$,

its electric dipole moment: $\vec{\mathcal{P}} = \int \varrho \vec{x} \, dv$,

its magnetic dipole moment: $\vec{\mathcal{M}} = \frac{1}{2} \int \vec{x} \wedge \vec{s} \, dv$,

its electric quadrupole moment: $Q_{ik} = \frac{1}{2} \int \varrho x_i x_k \, dv$.

Instead of (49) and (50) we may also write:

$$\int (\varrho \mathcal{B} + \vec{s}\vec{\mathcal{A}}) \, dv = \varepsilon \mathcal{B}_0 + \vec{\mathcal{E}}\vec{\mathcal{P}} + \vec{\mathcal{M}}\vec{\mathcal{H}} + (Q \, grad_0) \vec{\mathcal{E}} + \\ + \frac{d}{cdt} \{\vec{\mathcal{A}}\vec{\mathcal{P}} + (Q \, grad_0) \vec{\mathcal{E}}\}. \quad (51)$$

(For processes in which the total energy of the system is conserved the matrix elements of the time derivative on the right of (51) vanish).

b) Separation of the static meson field. The equations determining the static fields are:

$$\left.\begin{array}{lll}
\left.\begin{array}{l} \mathring{\vec{\mathbf{F}}} = -\operatorname{grad} \mathring{\mathbf{V}}, \\ \mathring{\vec{\mathbf{G}}} = \operatorname{rot} \mathring{\vec{\mathbf{U}}} + \vec{\mathbf{S}}, \end{array}\right\} & \mathring{\mathbf{f}}_{ik} = \mathring{\Phi}_{ik} + \mathring{\mathbf{S}}_{ik}, & \\
\left.\begin{array}{l} \mathring{\vec{\Gamma}} = -\operatorname{grad} \mathring{\Psi} + \vec{\mathbf{P}}, \\ \mathring{\Phi} = 0, \end{array}\right\} & \mathring{\mathbf{h}}_i = -\mathring{\mathbf{w}}_{|i} + \mathring{\mathbf{S}}_i^{(0)}, & \Bigg\} \; \mathring{\mathbf{f}}_{\mu\nu} = \mathring{\Phi}_{\mu\nu} + \mathring{\mathbf{S}}_{\mu\nu}. \\
\left.\begin{array}{l} \operatorname{div} \mathring{\vec{\mathbf{F}}} = -\mu^2 \mathring{\mathbf{V}} + \mathbf{N}, \\ \operatorname{rot} \mathring{\vec{\mathbf{G}}} = -\mu^2 \mathring{\vec{\mathbf{U}}}, \end{array}\right\} & \mathring{\mathbf{f}}^{ik}{}_{|k} = -\mu^2 \mathring{\mathbf{u}}^i + \mathring{\mathbf{M}}^i, & \\
\operatorname{div} \mathring{\vec{\Gamma}} = -\mu^2 \mathring{\Psi}, & \mathring{\mathbf{h}}^k{}_{|k} = -\mu^2 \mathring{\mathbf{w}} + \mathring{\mathbf{M}}^{(0)}, & \Bigg\} \; \mathring{\mathbf{f}}^{\mu\nu}{}_{|\nu} = -\mu^2 \mathring{\mathbf{u}}^\mu + \mathring{\mathbf{M}}^\mu.
\end{array}\right\} \quad (52)$$

The equations in the first column define the static field; those in the second one are the same but now written in tensor form, while in the third column the 5-tensor form is indicated. Any "tensor" labeled with ° is the same function of the static variables as the corresponding tensor in the former equations is of the ordinary variables. Of course this covariant form of the equations has no further meaning as the process of splitting off the static field is not invariant; it will be seen however that it is very useful to work with these equations in their 5-tensor form.

As has been shown in M.R. the static part of the fields may be separated from all variables by means of a canonical transformation:

$$\tilde{A} = S^{-1} A S, \tag{53}$$

where the unitary operator S which transforms the function $\tilde{A}$ of the old variables (from now one indicated by $\vec{\tilde{\mathbf{U}}}, \tilde{\mathbf{V}}, \ldots$) to the *same* function of the new variables ($\vec{\mathbf{U}}, \mathbf{V}, \ldots$) is given by

$$S = \exp. \frac{i}{h\,c} K,$$

$$K = \int dv\,[\mathring{\mathbf{f}}_{\mu 4}\,\mathbf{u}^{\mu} - \mathring{\mathbf{u}}^{\mu}\,\mathbf{f}_{\mu 4}] = \int dv\,[\vec{\mathring{\mathbf{F}}}\,\vec{\mathbf{U}} - \vec{\mathring{\mathbf{U}}}\,\vec{\mathbf{F}} + \mathring{\Psi}\,\Phi]. \tag{54}$$

The transformation of the terms of H which do not depend on the electromagnetic field has been treated in M.R. Further $\int H_{(e)}\,dv$ is of course not affected by this transformation, so we have to consider only more closely the Coulomb energy and the last two terms of (46), or their equivalent (51). Thus all depends on the transformation of $\tilde{\varrho}$ and $\vec{\tilde{s}}$.

First we will transform $\vec{\tilde{s}}_{\text{mes}}$ and $\tilde{\varrho}_{\text{mes}}$ and we will do this by making use of the 5-vector $\tilde{s}^{\mu}_{\text{mes}}$ (see (40)) from which both can be derived. As a consequence of (53) *)

$$\vec{s}^{\mu}_{\text{mes}} = \frac{e}{h\,c} (\tilde{\mathbf{f}}^{\nu\mu} \wedge \tilde{\mathbf{u}}_{\nu})_3 = S^{-1}\, s^{\mu}_{\text{mes}}\, S = \sum_0^{\infty} \frac{1}{l!} \left\{ \frac{i}{h\,c} K, s^{\mu} \right\}^l$$

$$= s^{\mu}_{\text{mes}} + s^{\mu}_{(1)} + s^{\mu}_{(2)} + \sum_3^{\infty} \frac{1}{l!} \left\{ \frac{i}{h\,c} K, s^{\mu} \right\}^l. \tag{55}$$

Terms for which $l \geqq 3$ need not be taken into account, as they are of higher order than the second in g_1 and g_2.

---

*) $\{A, B\}^l \equiv [A, [A, \ldots [A, B]] \ldots]$; l is the number of brackets.

We now introduce a set of variables marked with 1 that refer to free mesons. Of course:

$$\overset{1}{\vec{\mathbf{U}}} = \vec{\mathbf{U}} \;,\; \overset{1}{\vec{\mathbf{F}}} = \vec{\mathbf{F}} \;,\; \overset{1}{\Psi} = \Psi \;,\; \overset{1}{\Phi} = \Phi ,$$

while

$$\overset{1}{\mathbf{V}} = -\mu^{-2}\, div\, \vec{\mathbf{F}} \;,\; \overset{1}{\vec{\mathbf{G}}} = rot\, \vec{\mathbf{U}} \;,\; \overset{1}{\vec{\Gamma}} = -\,grad\, \Psi ,$$

so

$$\vec{\mathbf{G}} = \overset{1}{\vec{\mathbf{G}}} + \vec{\mathbf{S}} \;,\; \mathbf{V} = \overset{1}{\mathbf{V}} + \mu^{-2}\mathbf{N} \;,\; \vec{\Gamma} = \overset{1}{\vec{\Gamma}} + \vec{\mathbf{P}},$$

$$\vec{s}_{mes} = \overset{1}{\vec{s}}_{mes} + \frac{e}{h\,c}\,[\vec{\mathbf{S}} \wedge \vec{\mathbf{U}} - \mu^{-2}\vec{\mathbf{F}} \wedge \mathbf{N} + \vec{\mathbf{P}} \wedge \Psi]_3 . \qquad (56)$$

For the second term in the development we have

$$s^{\mu}_{(1)} = \frac{i}{h\,c}\,[K, s^{\mu}] .$$

In calculating this commutator we must use (43); we then get *)

$$s^{\mu}_{(1)} = \frac{e}{hc}\,\{\overset{\circ}{\mathbf{f}}_{\varrho 4} \wedge \mathbf{u}_{\nu})_3\, \gamma^{[\mu}_{\;\;.4}\, \pi^{\nu]\varrho,44} - (\overset{\circ}{\mathbf{u}}{}^{\varrho} \wedge \mathbf{f}^{\nu\mu})_3\, \pi_{\varrho\nu,4}^{\;\;\;\;\;\;.4}\} -$$

$$- \frac{e}{h\,c}\cdot\frac{i}{h\,c}\left\{\int (\overset{\circ}{\mathbf{f}}{}'_{\varrho 4} \wedge \mathbf{f}^{\nu\mu})_3 \,.\, [u'^{\varrho}, \overset{1}{u}_{\nu}]\, dv' +\right.$$

$$\left. + \int (\overset{\circ}{\mathbf{u}}{}'^{\varrho} \wedge \mathbf{u}_{\nu})_3\, [f'_{\varrho 4}, \overset{1}{f}{}^{\nu\mu}]\, dv' . \right. \qquad (57)$$

The commutators occurring in the integrals are composed of quantities with the same isotopic index, which has been omitted. The contribution of each term to the 4-vector $s^{i}_{(1)}$ is then found by inner multiplication with $\gamma_{\mu}^{\;.i}$ and with the help of (15*b*), (16*b*) and of I (16)—(18). Thus the first term contributes (apart from the factor e/hc)

$$\text{to } \varrho_{(1)}\!: \; -(\overset{\circ}{\vec{\mathbf{F}}} \wedge \vec{\mathbf{U}})_3 ; \;\; \text{to } \vec{s}_{(1)}\!: \; -(\overset{\circ}{\vec{\mathbf{F}}} \wedge \mathbf{V})_3 = -(\overset{\circ}{\vec{\mathbf{F}}} \wedge \overset{1}{\mathbf{V}})_3 - \mu^{-2}(\overset{\circ}{\vec{\mathbf{F}}} \wedge \mathbf{N})_3 ,$$

*) The first term has been computed making use of:

$$[u'^{\varrho}, f^{\nu\mu}] = \gamma^{\nu}_{\;.4}\,[u'^{\varrho}, f^{4\mu}] + \gamma^{\mu}_{\;.4}\,[u'^{\varrho}, f^{\nu 4}] .$$

and the second

to $\varrho_{(1)}$: $-(\vec{F} \wedge \overset{\circ}{\vec{U}})_3 - (\overset{\circ}{\Psi} \wedge \Phi)_3$;

to $\vec{s}_{(1)}$: $(\overset{1}{\vec{G}} \mathbin{\hat{\wedge}} \overset{\circ}{\vec{U}})_3 + (\overset{1}{\vec{\Gamma}} \wedge \overset{\circ}{\Psi})_3 + (\vec{S} \mathbin{\hat{\wedge}} \overset{\circ}{\vec{U}})_3 + (\vec{P} \wedge \overset{\circ}{\Psi})_3$.

The third term gives, after multiplication with $\gamma_{\mu}^{\cdot i}$, and keeping in mind that $\overset{\circ}{\mathbf{h}}{}^4 = \overset{\circ}{\Phi} = 0$

$$-\frac{i}{hc}\int (\overset{\circ}{\mathbf{f}}{}'_{l4} \wedge \mathbf{f}^{ki})_3 \, [u'^{\,l}, \overset{1}{u}_k],$$

so it does not contribute to $\varrho_{(1)}$ while it gives for $\vec{s}_{(1)}$

$$-\frac{i}{hc}\sum_{l=1}^{3}\int (\overset{\circ}{\vec{F}}{}'_l \wedge \vec{F})_3 \, [U'_l, \overset{1}{V}]\, dv' = \frac{i}{hc}\sum_{l=1}^{3}\int (\overset{\circ}{\vec{F}}{}'_l \wedge \vec{F})_3 \, [U'_l, div\, \vec{F}].$$

Now

$$[U'_l, div\, \vec{F}] = \frac{hc}{i}\frac{\partial}{\partial x^l}\,\delta(\vec{x}-\vec{x}') = -\frac{hc}{i}\frac{\partial}{\partial x'^{\,l}}\,\delta(\vec{x}-\vec{x}'),$$

therefore the third term gives for $\vec{s}_{(1)}$ after a partial integration

$$-(\vec{F} \wedge \overset{\circ}{V})_3 + \mu^{-2}(\vec{F} \wedge N)_3 .$$

Finally the last term of (57) becomes after multiplying with $\gamma_{\mu}^{\cdot i}$

$$-\frac{i}{hc}\left\{\int (\overset{\circ}{\mathbf{u}}{}'^k \wedge \mathbf{u}_l)_3 \, [f'_{k4}, \overset{1}{f}{}^{li}] + \int (\overset{\circ}{\mathbf{w}}{}' \wedge \mathbf{w})_3 \, [h'_4, \overset{1}{h}{}^i]\right\}$$

from which we infer that it does not contribute to $\varrho_{(1)}$. The contribution to $\vec{s}_{(1)}$ can be found in a similar way to the treatment of the preceding term. The result is

$$(\overset{\circ}{\vec{G}} \mathbin{\hat{\wedge}} \vec{U})_3 + (\overset{\circ}{\vec{\Gamma}} \wedge \Psi)_3 - (\vec{S} \mathbin{\hat{\wedge}} \vec{U})_3 - (\vec{P} \wedge \Psi)_3.$$

Consequently the complete result of (57) becomes

$$\varrho_{(1)} \equiv \varrho_{\times} = \frac{e}{hc}(\overset{\circ}{\vec{U}} \wedge \vec{F} + \vec{U} \wedge \overset{\circ}{\vec{F}} - \overset{\circ}{\Psi} \wedge \Phi)_3, \qquad (58)$$

$$\vec{s}_{(1)} = \vec{s}_{\times} - \frac{e}{hc} (\vec{\mathrm{S}} \mathbin{\overset{\wedge}{\wedge}} \vec{\mathrm{U}} - \mu^{-2} \vec{\mathrm{F}} \wedge \mathrm{N} + \vec{\mathrm{P}} \wedge \Psi)_3 +$$

$$+ \frac{e}{hc} (\vec{\mathrm{S}} \mathbin{\overset{\wedge}{\wedge}} \overset{\circ}{\vec{\mathrm{U}}} - \mu^{-2} \overset{\circ}{\vec{\mathrm{F}}} \wedge \mathrm{N} + \vec{\mathrm{P}} \wedge \overset{\circ}{\vec{\Psi}})_3, \quad (59)$$

where

$$\vec{s}_{\times} = \frac{e}{hc} (\overset{\circ}{\vec{\mathrm{G}}} \mathbin{\overset{\wedge}{\wedge}} \vec{\mathrm{U}} + \overset{1}{\vec{\mathrm{G}}} \mathbin{\overset{\wedge}{\wedge}} \overset{\circ}{\vec{\mathrm{U}}} - \overset{\circ}{\vec{\mathrm{F}}} \wedge \overset{1}{\mathrm{V}} - \vec{\mathrm{F}} \wedge \overset{\circ}{\mathrm{V}} + \overset{\circ}{\vec{\Gamma}} \wedge \Psi + \overset{1}{\vec{\Gamma}} \wedge \overset{\circ}{\Psi})_3. \quad (60)$$

Now we must find the third term of (55)

$$s^{\mu}_{(2)} = \frac{i}{2hc} [K, s^{\mu}_{(1)}].$$

The calculation of this commutator goes by the same methods which we have used to find $s^{\mu}_{(1)}$. It gives rise to field-independent (f.i.) terms as well as to terms quadratic in the meson field components. For the calculations in the following chapter we are only interested in the former which we here directly give:

$$\text{f.i. part of } \varrho_{(2)} \equiv \varrho_{\mathrm{exch}} = \frac{e}{hc} (\overset{\circ}{\vec{\mathrm{U}}} \wedge \overset{\circ}{\vec{\mathrm{F}}})_3\,, \quad (61)$$

$$\text{f.i. part of } \vec{s}_{(2)} = \vec{s}_{\mathrm{exch}} - \frac{e}{hc} (\vec{\mathrm{S}} \mathbin{\overset{\wedge}{\wedge}} \overset{\circ}{\vec{\mathrm{U}}} - \mu^{-2} \overset{\circ}{\vec{\mathrm{F}}} \wedge \mathrm{N} + \vec{\mathrm{P}} \wedge \overset{\circ}{\Psi})_3\,, \quad (62)$$

with

$$\vec{s}_{\mathrm{exch}} = \frac{e}{hc} (\overset{\circ}{\vec{\mathrm{G}}} \mathbin{\overset{\wedge}{\wedge}} \overset{\circ}{\vec{\mathrm{U}}} - \overset{\circ}{\vec{\mathrm{F}}} \wedge \overset{\circ}{\mathrm{V}} + \overset{\circ}{\vec{\Gamma}} \wedge \overset{\circ}{\Psi})_3\,. \quad (63)$$

Thus, from (58) and (61) we may infer that to the approximation indicated

$$\tilde{\varrho}_{\mathrm{mes}} = \varrho_{\mathrm{mes}} + \varrho_{\times} + \varrho_{\mathrm{exch}} \quad (64)$$

and similarly from (56), (59) and (62) that

$$\tilde{\vec{s}}_{\mathrm{mes}} = \overset{1}{\vec{s}}_{\mathrm{mes}} + \vec{s}_{\times} + \vec{s}_{\mathrm{exch}}\,. \quad (65)$$

(64) and (65) can immediately be understood if one remembers that any fieldvariable $\tilde{A}$ occurring in $\tilde{\varrho}_{\mathrm{mes}}$ is approximately the

sum of a static part $\mathring{A}$ and a new free meson variable A. Inserting this one gets: *a*) a part of $\tilde{\varrho}_{\text{mes}}$ depending only on the free meson variables, namely ϱ_{mes}; *b*) a part depending both on A and $\mathring{A}$: $\varrho_{\times}$; *c*) field-independent part of $\tilde{\varrho}_{\text{mes}}$: ϱ_{exch}! Similarly for $\tilde{\vec{s}}_{\text{mes}}$.

We must now apply the same transformation to $\tilde{s}^{\mu}_{\text{nucl}}$. This calculation is quite straightforward. For the developments of the next chapter we only need the f.i. part of $\tilde{s}^{\mu}_{\text{nucl}}$ and it can be seen that this is simply s^{μ}_{nucl}. Thus, summarizing the result of the transformation, we may state that, in order to compute $\vec{\mathfrak{P}}$, $\vec{\mathfrak{M}}$ and Q in our approximation, we may put

$$\tilde{\varrho} = \varrho_{\text{nucl}} + \varrho_{\text{exch}},$$

$$\tilde{\vec{s}} = \vec{s}_{\text{nucl}} + \vec{s}_{\text{exch}}.$$

To calculate $\vec{\mathfrak{P}}$ we remark that $\mathring{\vec{\mathbf{F}}} = -grad\ \mathring{\mathbf{V}}$ and

$$\int \Phi^{(i)}\,\Phi^{(k)}\,dv = \frac{e^{-\mu r_{ik}}}{8\pi\mu}; \qquad \Phi^{(i)} = \frac{1}{4\pi\,|\vec{x}-\vec{x}^{(i)}|}\,\exp.-\varkappa\,|\vec{x}-\vec{x}^{(i)}|.$$

Bringing $\mathring{\vec{\mathbf{U}}}$ and $\mathring{\mathbf{V}}$ in a form in which the nuclear variables occur explicitly (cf. M.R. equ. (14)) we get:

$$\vec{\mathfrak{P}} = \frac{e}{2}\sum_i (1-\tau_3^{(i)})\,\vec{x}^{(i)} - \frac{e}{8\pi hc}\cdot\frac{g_1 g_2}{\varkappa}\sum_{i,k}(\boldsymbol{\tau}^{(i)}\wedge\boldsymbol{\tau}^{(k)})_3\,.\,(\vec{\sigma}^{(i)}\wedge\vec{x}_0^{ik})\,.\,e^{-\mu r_{ik}}, \qquad (66)$$

with

$$\vec{x}_0^{ik} = \frac{\vec{x}^{(i)}-\vec{x}^{(k)}}{|\vec{x}^{(i)}-\vec{x}^{(k)}|}.$$

Further we have replaced $\varrho_3^{(i)}$ by 1 which only gives a difference of the second order in the velocities.

Q is found in the same way. Here we make use of

$$\int \vec{x}\,\Phi^{(i)}\,\Phi^{(k)}\,dv = \frac{e^{-\mu r_{ik}}}{8\pi\mu}\;\frac{\vec{x}^{(i)}+\vec{x}^{(k)}}{2}$$

and get

$$Q_{mn} = \frac{e}{2} \sum_i (1-\tau_3^{(i)})\, x_m^{(i)}\, x_n^{(i)} -$$

$$-\frac{e}{16\pi hc}\cdot\frac{g_1 g_2}{\varkappa} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \,.\, (x_m^{(i)} + x_m^{(k)}) \,.\, (\vec{\sigma}^{(i)} \wedge \vec{x}_0^{ik})_n\, e^{-\mu r_{ik}} \,. \quad (67)$$

For $\vec{\mathfrak{M}}$ we have

$$\vec{\mathfrak{M}} = \vec{\mathfrak{M}}_{\text{nucl}} + \vec{\mathfrak{M}}_{\text{exch}}\,.$$

To compute $\vec{\mathfrak{M}}_{\text{exch}}$ we introduce in $\vec{s}_{\text{exch}}$ the explicit expressions for the static field variables (see also M.R. (37) and (40)) and then get

$$\vec{s}_{\text{exch}} = \frac{e}{hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \Big[-g_2^2\, \vec{\sigma}^{(i)}\, \Phi^{(i)} \wedge (\vec{\sigma}^{(k)} \wedge \vec{f}^{(k)}) +$$

$$+ \frac{g_2^2}{\mu^2} (\vec{\sigma}^{(i)}\, \vec{\nabla}^{(i)})\, \vec{f}^{(i)} \wedge (\vec{\sigma}^{(k)} \wedge \vec{f}^{(k)}) - g_1^2\, \vec{f}^{(i)}\, \Phi^{(k)} +$$

$$+ \frac{g_2^2}{\mu^2}\, \vec{\sigma}^{(i)}\, (\vec{\sigma}^{(k)}\, \vec{f}^{(k)})\, \delta\, (\vec{x} - \vec{x}^{(i)}) + \frac{g_2^2}{\mu^2} \{ (\vec{\sigma}^{(i)}\, \vec{\nabla}^{(i)})\, \vec{f}^{(i)} \}\, (\vec{\sigma}^{(k)}\, \vec{f}^{(k)}) \Big],$$

$$\vec{f}^{(k)} = grad^{(k)}\, \Phi^{(k)}, \quad \vec{\nabla}^{(i)} = grad^{(i)}\,.$$

Therefore $\vec{\mathfrak{M}}_{\text{exch}}$ becomes

$$\vec{\mathfrak{M}}_{\text{exch}} = \frac{e}{2\,hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \Big[-g_2^2 \int (\vec{x} \wedge \vec{\sigma}^{(k)})\, (\vec{\sigma}^{(i)}\, \vec{f}^{(k)})\, \Phi^{(i)}\, dv +$$

$$+ \frac{g_2^2}{\mu^2} \int (\vec{\sigma}^{(i)}\, \vec{\nabla}^{(i)})\, (\vec{f}^{(i)}\, \vec{f}^{(k)})\, (\vec{x} \wedge \vec{\sigma}^{(k)})\, dv + \frac{g_2^2}{\mu^2} (\vec{x}^{(i)} \wedge \vec{\sigma}^{(i)}) \{ \vec{\sigma}^{(k)}\, \vec{\nabla}_i^{(k)}\, \Phi\, (r_{ik}) \} +$$

$$+ \frac{g_2^2}{\mu^2} \int (\vec{\sigma}^{(i)} \wedge \vec{f}^{(i)})\, (\vec{\sigma}^{(k)}\, \vec{f}^{(k)})\, dv \Big].$$

The first two terms describe the "exchange" part of the magnetic dipole moment due to the vector field and the other two the contribution of the pseudoscalar field. The separate expressions for these two parts are:

$$\vec{\mathfrak{M}}_{\text{exch}}^{\text{vector}} = g_2^2 \cdot \frac{e}{2hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \left[(\vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)}) \left(\frac{1}{2\mu^2} - \frac{r_{ik}}{2\mu} \right) + \right.$$

$$\left. + (\vec{\sigma}^{(i)} \vec{r}_{ik}) \{ \vec{\sigma}^{(k)} \wedge (\vec{x}^{(i)} + \vec{x}^{(k)}) \} \left(\frac{1}{2\mu r_{ik}} + \frac{1}{2(\mu r_{ik})^2} \right) \right] \Phi(r_{ik}).$$

$$\vec{\mathfrak{M}}_{\text{exch}}^{\text{ps.sc}} = g_2^2 \cdot \frac{e}{2hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \left[(\vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)}) \cdot \frac{1}{2\mu^2} + \right.$$

$$\left. + (\vec{\sigma}^{(i)} \vec{r}_{ik}) \{ \vec{\sigma}^{(k)} \wedge (\vec{x}^{(i)} - 3\vec{x}^{(k)}) \} \left(\frac{1}{2\mu r_{ik}} + \frac{1}{2(\mu r_{ik})^2} \right) \right] \Phi(r_{ik}).$$

These expressions depend on the coordinates of the centre of gravity of the *i–th* and *k–th* nucleon (neglecting the difference between m_N and m_P) and it is remarkable that this is no more the case for $\vec{\mathfrak{M}}_{\text{exch}} = \vec{\mathfrak{M}}_{\text{exch}}^{\text{vector}} + \vec{\mathfrak{M}}_{\text{exch}}^{\text{ps.sc}}$, which, as regards spatial variables, only depends on $\vec{r}_{ik}$, and thus is a translation-invariant quantity. The complete expression for the magnetic dipole moment of a nuclear system thus is

$$\vec{\mathfrak{M}} = \frac{e}{2} \sum_i \frac{1-\tau_3^{(i)}}{2} (\vec{x}^{(i)} \wedge \vec{\alpha}^{(i)}) + g_2^2 \cdot \frac{e}{2hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 \left[(\vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)}) \left(\frac{1}{\mu^2} - \frac{r_{ik}}{2\mu} \right) + \right.$$

$$\left. + (\vec{\sigma}^{(i)} \vec{r}_{ik}) (\vec{\sigma}^{(k)} \wedge \vec{r}_{ik}) \left(\frac{1}{\mu r_{ik}} + \frac{1}{(\mu r_{ik})^2} \right) \right] \Phi(r_{ik}).$$

Finally, we will give here for later purposes the expression for the time derivative of $\vec{\mathfrak{P}}$ which has been computed by MøLLER and ROSENFELD [10]):

$$\dot{\vec{\mathfrak{P}}} = \frac{e}{2} \sum_i (1-\tau_3^{(i)}) \vec{\alpha}^{(i)} +$$

$$+ \frac{e}{2hc} \sum_{i,k} (\tau^{(i)} \wedge \tau^{(k)})_3 (\vec{x}^{(i)} - \vec{x}^{(k)}) (g_1^2 + g_2^2 \vec{\sigma}^{(i)} \vec{\sigma}^{(k)}) \Phi(r_{ik}).$$

In calculating the time derivative of the quadrupole moment we have made use of the same method which was followed to find $\dot{\vec{\mathfrak{P}}}$. We will here merely state the result:

$$\dot{Q}_{ik} = \frac{e}{4} \sum_j (1 - \tau_3^{(j)}) (\vec{a}_i^{(j)} \vec{x}_k^{(j)} + \vec{x}_i^{(j)} \vec{a}_k^{(j)}) +$$

$$+ \frac{e}{4hc} \sum_{m,n} (\tau^{(m)} \wedge \tau^{(n)})_3 (x_i^{(m)} x_k^{(m)} - x_i^{(n)} x_k^{(n)}) (g_1^2 + g_2^2 \vec{\sigma}^{(m)} \vec{\sigma}^{(n)}) \, \Phi \, (r_{mn}).$$

REFERENCES.

1) H. YUKAWA, Proc. phys. math. Soc. Japan, **17**, 48, 1935.
2) N. KEMMER, Proc. roy. Soc. A, **166**, 154, 1938.
3) N. KEMMER, Proc. Cambridge Phil. Soc. **34**, 358, 1938.
4) H. A. BETHE, Phys. Rev. **55**, 1261, 1939.
5) C. MøLLER and L. ROSENFELD, D. Danske Vid. Selsk. math.-fys. Medd., **17**, fasc. 8, 1940.
6) C. MøLLER, D. Danske Vid. Selsk. math.-fys. Medd., **18**, fasc. 6, 1941.
7) J. A. SCHOUTEN, Ann. Inst. H. Poincaré **5**, 49, 1935.
8) H. J. BHABHA, Proc. roy. Soc. A **166**, 501, 1938.
9) S. ROZENTAL, *in the press.*
10) C. MøLLER and L. ROSENFELD, *in course of publication.*

CHAPTER III.

THE PHOTO-EFFECT OF THE DEUTERON.

Summary.

The photo-disintegration of the deuteron and the capture of neutrons by protons are discussed from the standpoint of the MøLLER–ROSENFELD theory of nuclear forces. The general expression for the cross section of the photo-electric effect turns out to be identical in form with the corresponding quantity in the old BETHE–PEIERLS theory, while the photo-magnetic cross section contains an extra term due to the meson field. As a consequence of the different wave functions used for the ground state of the deuteron the cross sections decrease more rapidly with increasing photon energy than in the old theory. The absolute values for the cross sections are of the same order of magnitude as found empirically, though definite numerical results can as yet not be given, owing to the unreliability of the deuteron wave functions used. This circumstance makes a definite statement with regard to the angular distribution premature. The capture cross sections also are of the right order of magnitude and, as in the old theory, the $1/v$ law appears to be a magnetic effect.

§ 1. *Introduction.* The discovery, made by CHADWICK and GOLDHABER [1]), that the deuteron can be disintegrated under the influence of γ-rays of sufficiently high energy, provides us with most valuable information about the interaction of electromagnetic radiation with nuclear systems. This effect is closely connected with the capture process of neutrons by protons, which especially plays a prominent rôle in experiments with slow neutrons. In the earliest treatments that were given of the photo disintegration [2]) [3]) as well as of the capture, these effects were considered as photo-electric (PE) processes, (interaction of the electric field of the incident wave with the nuclear system). The cross sections thus obtained for the PE disintegration were in reasonable agreement with experiment, but there turned out to be a difference of several orders of magnitude between theoretical expectations and the measured values of the capture cross section. This point was cleared up by the remark of FERMI [4]) that, besides the mentioned processes, one has also to take into account the photomagnetic (PM) transitions, due to the interaction of the magnetic field of the incident wave with the magnetic moments of the nuclear particles, (cf. also BREIT and CONDON [5])); it was shown by him that the slow neutron capture is essentially of magnetic character and that the well-known $1/v$ law could be explained on this assumption. Thus, all experimental data known at the time could be accounted for on a theory based only on the assumption that the range of the nuclear forces was small compared to the radius of the electron.

More recent experiments by VON HALBAN [6]), however, seem to indicate a discrepance with theoretical expectations on the angular distribution of the disintegration products: while for the PM effect, (corresponding to a transition between the ^3S-state and the ^1S-state of the deuteron), this distribution is isotropic, the contribution of the PE effect (a $^3S \rightarrow {}^3P$ transition) per unit solid angle is proportional to $\sin^2\Theta$, Θ being the angle between the incident γ-ray and the ejected neutron. Therefore, from the expressions for the differential cross section of both effects, which we will call $d\Phi^{el}(\Theta)$ and $d\Phi^{magn}$, we find for the ratio of the intensities at $\Theta = 0$, ($\Phi_{||}$) and $\Theta = \pi/2$, ($\Phi_{\perp}$):

$$\frac{\Phi_{||}}{\Phi_{\perp}} = \frac{d\,\Phi^{magn}}{d\,\Phi^{magn} + d\,\Phi^{el}(\pi/2)}. \tag{1}$$

For ThC'' γ-rays this ratio was calculated to be 0,29 (assuming the ^1S-level of the deuteron to be a virtual one, as now seems to be certain), while the measurements of VON HALBAN give a value that, (considering the experimental uncertainties), lies between 0,01 and 0,13. This effect, if real, constitutes a difficulty which may be expected to be cleared up only by a deeper insight in the nature of nuclear forces. It is therefore of interest to see whether our present conceptions of the interaction between nucleons can clarify this point.

We shall here for this purpose adopt the standpoint of the theory of MøLLER and ROSENFELD [7]), according to which nuclear forces are described by a mixture of vector and pseudoscalar meson fields *). In the next two sections we will treat the PE and PM

*) Recently, a discussion of the *PE* effect in the frame of the meson theory of nuclear forces was given by FRÖHLICH, HEITLER and KAHN [8]), assuming the interaction to be described by a field of the vector type. However, their "Ansatz" is clearly inconsistent with the general electromagnetic properties of nuclear systems; we will therefore here not consider their results.

effect from this point of view, while in the last section the capture problem has been dealt with. The admixture of a D-state with the ^3S-ground state of the deuteron has, on account of its smallness, practically no influence on the effects under consideration. We will therefore throughout neglect the contribution of this D-state, and thus consider the ground state to be purely of the ^3S type.

§ 2. a) *The wave equation of the deuteron.* We will first give a survey of the properties of the deuteron wave functions, representing in a slightly different form results obtained by KEMMER [9]) in a paper on the neutron-proton interaction.

The two nucleons that constitute the deuteron, and all quantities that refer to them, are labeled with the upper indices 1 and 2 respectively; thus, for instance, $\vec{x}^{(1)}$, $\vec{p}^{(1)}$ and $\vec{x}^{(2)}$, $\vec{p}^{(2)}$ represent the spatial coordinates and impulse, (multiplied by the velocity of light c), of the first and second particle. The deuteron is described by a 16-component wave function $\Psi_{E,\mu}$, (μ stands for all those sets of values of the degeneracy parameters that belong to the same energy E). In the frame of reference in which the centre of gravity of the deuteron is at rest it satisfies the equation *):

$$H_0\, \Psi_{E\mu}(\vec{x}) = \left(\frac{hc}{i}\, \vec{\alpha}\, \mathrm{grad} + \beta\, M c^2 + V(r)\right) \Psi_{E\mu}(\vec{x}) = E\, \Psi_{E\mu}(\vec{x}) \quad (2)$$

with

$$\vec{x} = \vec{x}^{(1)} - \vec{x}^{(2)},\ \ r = |\vec{x}|,\ \ \vec{\alpha} = \vec{\alpha}^{(1)} - \vec{\alpha}^{(2)},\ \ \beta = \varrho_3^{(1)} + \varrho_3^{(2)},\ \ M \cong M_N \cong M_P,$$

$$V = (\tau^{(1)}\, \tau^{(2)})\, [g_1^2 + g_2^2\, \vec{\sigma}^{(1)}\, \vec{\sigma}^{(2)}] \cdot \frac{e^{-\varkappa r}}{4\pi r}.$$

According to KEMMER we can classify the non-trivial proper solutions of (2) as follows:

Type Ia:	corresponds in non-relativistic approximation with	triplet state with $l = j \pm 1$.
Type Ib:		triplet state with $l = j$.
Type IIb:		singlet state, $(l = j)$.

*) h is PLANCK's constant divided by 2π.

Next we introduce the normalized spin wave functions:

$$\left.\begin{aligned} {}^3\chi_0 &= \frac{1}{\sqrt{2}}\left(\delta_{\sigma_3^{(1)},1}\,\delta_{\sigma_3^{(2)},-1} + \delta_{\sigma_3^{(1)},-1}\,\delta_{\sigma_3^{(2)},1}\right), \\ {}^3\chi_1 &= \delta_{\sigma_3^{(1)},1}\,\delta_{\sigma_3^{(2)},1} \qquad {}^3\chi_{-1} = \delta_{\sigma_3^{(1)},-1}\,\delta_{\sigma_3^{(2)},-1}, \\ {}^1\chi_0 &= \frac{1}{\sqrt{2}}\left(\delta_{\sigma_3^{(1)},1}\,\delta_{\sigma_3^{(2)},-1} - \delta_{\sigma_3^{(1)},-1}\,\delta_{\sigma_3^{(2)},1}\right), \end{aligned}\right\} \tag{3}$$

and similarly the "isotopic spin wave functions" ζ and the "ϱ-wave functions" ξ:

$$^3\zeta_1 = \delta_{\tau_3^{(1)},1}\,\delta_{\tau_3^{(2)},1}, \text{ etc.}; \tag{3a}$$

$$^3\xi_1 = \delta_{\varrho_3^{(1)},1}\,\delta_{\varrho_3^{(2)},1}, \text{ etc.} \tag{3b}$$

We then have, to the first order in the velocities, ($\overset{(0)}{\Psi}$ is the velocity-independent part of Ψ which we will call "large component", $\overset{(1)}{\Psi}$ which is of the order of v/c is the "small component")

$$\Psi = \overset{(0)}{\Psi} + \overset{(1)}{\Psi}, \tag{4}$$

with

$$\left.\begin{aligned} &\text{Type I}a, \text{I}b: \ \overset{(0)}{\Psi} = {}^3\xi_1\,[{}^3\chi_1\,Z^1 + {}^3\chi_0\,Z^0 + {}^3\chi_{-1}\,Z^{-1}], \\ &\text{Type II}b \quad : \ \overset{(0)}{\Psi} = {}^3\xi_1\,{}^1\chi_0\,Z_0, \end{aligned}\right\} \tag{5a}$$

and

$$\left.\begin{aligned} &\text{Type I}a, \text{I}b: \ \overset{(1)}{\Psi} = {}^1\xi_0\,[{}^3\chi_1\,z^1 + {}^3\chi_0\,z_s^0 + {}^3\chi_{-1}\,z^{-1}] + {}^3\xi_0\,{}^1\chi_0\,z_a^0, \\ &\text{Type II}b \quad : \ \overset{(1)}{\Psi} = {}^3\xi_0\,[{}^3\chi_1\,z_1 + {}^3\chi_0\,z_s^0 + {}^3\chi_{-1}\,z^{-1}]. \end{aligned}\right\} \tag{5b}$$

The functions Z and z only depend on the relative spatial coordinates; introducing polar variables, ($x = r \sin\vartheta \cos\Phi$, $y = r \sin\vartheta \sin\Phi$, $z = r\cos\vartheta$), we obtain the following expressions for them, (to these we will refer as (6)):

$$\text{Type Ia, } l=j-1 \left\{ \begin{array}{ll} Z^1 = \sqrt{(j+m-1)(j+m)} & Y_{j-1}^{(m-1)} \\ Z^0 = -\sqrt{2(j+m)(j-m)} & Y_{j-1}^{(m)} \\ Z^{-1} = \sqrt{(j-m-1)(j-m)} & Y_{j-1}^{(m+1)} \end{array} \right\} \cdot \frac{1}{\sqrt{2j(2j-1)}} \cdot \frac{R_a(j-1)}{r}.$$

$$\text{Type Ia, } l=j+1 \left\{ \begin{array}{ll} Z^1 = \sqrt{(j-m+1)(j-m+2)} & Y_{j+1}^{(m-1)} \\ Z^0 = \sqrt{2(j+m+1)(j-m+1)} & Y_{j+1}^{(m)} \\ Z^{-1} = \sqrt{(j+m+1)(j+m+2)} & Y_{j+1}^{(m+1)} \end{array} \right\} \cdot \frac{1}{\sqrt{2(j+1)(2j+3)}} \cdot \frac{R_a(j+1)}{r}.$$

$$\text{Type Ib} \left\{ \begin{array}{ll} Z^1 = -\sqrt{(j+m)(j-m+1)} & Y_{j}^{(m-1)} \\ Z^0 = -m\sqrt{2} & Y_{j}^{(m)} \\ Z^{-1} = \sqrt{(j+m+1)(j-m)} & Y_{j}^{(m+1)} \end{array} \right\} \cdot \frac{1}{\sqrt{2j(j+1)}} \cdot \frac{R_b(j)}{r}.$$

$$\text{Type IIb } Z^0 = Y_j^{(m)} \quad \frac{R_{II}(j)}{r}.$$

$$\text{Type Ia, } l=j-1 \left\{ \begin{array}{ll} z^1 = -\sqrt{2(j+m)(j-m+1)} & Y_{j}^{(m-1)} \\ z_s^0 = -2m & Y_{j}^{(m)} \\ z^{-1} = \sqrt{2(j+m+1)(j-m)} & Y_{j}^{(m+1)} \\ z_a^0 = -2j & Y_j^{(m)} \end{array} \right\} \cdot \frac{1}{2\sqrt{2j}} \cdot \frac{C_0(j-1)}{r}.$$

$$\text{Type Ia, } l=j+1 \left\{ \begin{array}{ll} z^1 = -\sqrt{2(j+m)(j-m+1)} & Y_{j}^{(m-1)} \\ z_s^0 = -2m & Y_{j}^{(m)} \\ z^{-1} = \sqrt{2(j+m+1)(j-m)} & Y_{j}^{(m+1)} \\ z_a^0 = 2(j+1) & Y_j^{(m)} \end{array} \right\} \cdot \frac{1}{2\sqrt{2(j+1)}} \cdot \frac{C_2(j+1)}{r}$$

$$\text{Type*)} \begin{array}{l} \text{Ib} \\ \text{IIb} \end{array} \left\{ \begin{array}{ll} z^1 = \sqrt{2(j+m-1)(j+m)} & Y_{j-1}^{(m-1)} \\ z_s^0 = -2\sqrt{(j+m)(j-m)} & Y_{j-1}^{(m)} \\ z^{-1} = \sqrt{2(j-m-1)(j-m)} & Y_{j-1}^{(m+1)} \\ z_a^0 = 0. & \end{array} \right\} \cdot \begin{array}{l} \dfrac{1}{2\sqrt{2j(j+1)(2j-1)}} \cdot \dfrac{C^1(j)}{r} + \\ \dfrac{1}{2\sqrt{2(2j-1)}} \cdot \dfrac{H^1(j)}{r} \end{array}$$

$$+ \left\{ \begin{array}{ll} \sqrt{2(j-m+1)(j-m+2)} & Y_{j+1}^{(m-1)} \\ 2\sqrt{(j+m+1)(j-m+1)} & Y_{j+1}^{(m)} \\ \sqrt{2(j+m+1)(j+m+2)} & Y_{j+1}^{(m+1)} \end{array} \right\} \cdot \begin{array}{l} \dfrac{1}{2\sqrt{2j(j+1)(2j+3)}} \cdot \dfrac{C^2(j)}{r} \\ \dfrac{1}{2\sqrt{2(2j+3)}} \cdot \dfrac{H^2(j)}{r} \end{array}$$

---

*) The expressions before the braces are the same for both types. The upper expressions behind the braces refer to I*b*-states, the lower to II*b*-states.

The spherical harmonics are defined as in M.R. equ. (115); we also use the same normalization prescriptions as stated there.

The large radial functions $R(j)$ satisfy

$$\left\{\frac{h^2}{M}\left(\frac{d^2}{dr^2}-\frac{j(j+1)}{r^2}\right)+\Gamma.\frac{e^{-\varkappa r}}{r}+E\right\}R(j,r)=0, \tag{7}$$

where

$$\left.\begin{aligned} &\text{for type I}a,\ \text{I}b:\ \Gamma=[1-2(-1)^{j+1}].\frac{g_1^2+g_2^2}{4\pi},\\ &\text{for type II}b:\ \Gamma=[1-2(-1)^{j}]\quad.\frac{g_1^2-3g_2^2}{4\pi}.\end{aligned}\right\} \tag{7a}$$

From (7) it follows that R_∞, the asymptotic solution for R, is given by

$$R_\infty=\lambda\sqrt{\frac{2}{\pi}}\cos(kr+\varepsilon_j);\quad \varepsilon_j=-\frac{\pi}{2}(j+1)+\delta j;\quad k=\sqrt{ME}/h \tag{8}$$

The factor $\lambda\sqrt{2/\pi}$ normalizes R_∞ in the energy scale; as shown by BETHE and BACHER [10]) we have for λ

$$\lambda=\sqrt{\frac{1}{hv}}=\sqrt{\frac{M}{2h^2k}}. \tag{9}$$

The phase constants δ_j are essentially fixed by the solution of (7). With the exception of δ_0 they are negligibly small if $(h/\mathrm{ME})^{1/2}\gg\varkappa^{-1}$, (cf. BETHE and BACHER [10]), p. 115).

On account of (8) we may write for the asymptotic solution of the complete large wave function

$$\Psi_{E,\mu}(r\rightarrow\infty)=B_\zeta(E,\mu;\vartheta,\varphi).\frac{1}{r}\cos(kr+\varepsilon_j),\quad(\Psi\text{ large}). \tag{10a}$$

$$\zeta\equiv(\tau_3^{(1)},\tau_3^{(2)},\sigma_3^{(1)},\sigma_3^{(2)},\varrho_3^{(1)},\varrho_3^{(2)}),$$

Further, the small radial wave functions are related with the corresponding large functions by

$$\left.\begin{aligned}
C_0(j-1) &= \qquad \left(\frac{d}{dr}-\frac{j}{r}\right) R_a(j-1)\\
C_2(j+1) &= \qquad \left(\frac{d}{dr}+\frac{j+1}{r}\right) R_a(j+1)\\
C^1(j) &= (j+1)\cdot\left(\frac{d}{dr}+\frac{j}{r}\right) R_b(j)\\
C^2(j) &= \quad j\cdot\left(\frac{d}{dr}-\frac{j+1}{r}\right) R_b(j)\\
H^1(j) &= \quad -\left(\frac{d}{dr}+\frac{j}{r}\right) R_{II}(j)\\
H^2(j) &= \qquad \left(\frac{d}{dr}-\frac{j+1}{r}\right) R_{II}(j)
\end{aligned}\right\} \cdot \frac{ih}{Mc}\cdot\frac{1}{\sqrt{2j+1}} \qquad (11)$$

Similarly to (10) we can write for the asymptotic expression of the complete small wave function:

$$\Psi_{E\mu}(r \rightarrow \infty) = B_{\xi} \cdot \frac{1}{r} \cdot i \sin(kr + \varepsilon_j), \qquad (\Psi \text{ small}). \qquad (10b)$$

b) Interaction with electro-magnetic radiation *). We now examine the result of an irradiation of the deuteron with a monochromatic polarized γ-ray beam, and thus have to insert the operator Ω into the time dependent SCHROEDINGER equation of the deuteron:

$$ih\frac{\partial \Psi}{\partial t} = (H_0 + \Omega\, e^{-i\nu t} + \text{conj})\, \Psi,$$

where (see chapter II)

$$\Omega = \vec{\mathcal{E}}\vec{\mathfrak{P}} + \vec{\mathfrak{M}}\vec{\mathcal{H}} + (Q\, grad)\, \vec{\mathcal{E}}; \qquad (12)$$

$\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are the electric and magnetic amplitude respectively, taken at the centre of gravity of the system. $\vec{\mathfrak{P}}$, $\vec{\mathfrak{M}}$ and Q of course also

---

*) The developments in this section are analogous to the treatment of the *PE* effect for the hydrogen atom as given by BETHE [11]).

refer to the system in which the centre of gravity is at rest. Thus in the expressions for these quantities given in chapter II we must replace $\vec{x}^{(1)}$ by $\frac{\vec{x}}{2}$ and $\vec{x}^{(2)}$ by $-\frac{\vec{x}}{2}$. Consequently we have

$$\vec{\mathfrak{P}} = -\frac{e}{4}(\tau_3^{(1)} - \tau_3^{(2)})\vec{x} - \frac{e}{8\pi hc} \cdot \frac{g_1 g_2}{\varkappa}(\tau^{(1)} \wedge \tau^{(2)})_3 \cdot (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \wedge \vec{x}^0 . e^{-\varkappa r}, \quad (13)$$

$$\begin{aligned} \vec{\mathfrak{M}} = \frac{e}{8}\vec{x} \wedge \{(1-\tau_3^{(1)})\vec{\alpha}^{(1)} - (1-\tau_3^{(2)})\vec{\alpha}^{(2)}\} + \\ + g_2^2 \cdot \frac{e}{2hc} \sum_{i,k=1,2} (\tau^{(i)} \wedge \tau^{(k)})_3 \left[(\vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)}) \cdot \left(\frac{1}{\varkappa^2} - \frac{r_{ik}}{2\varkappa}\right) + \right. \quad (14) \\ \left. + (\vec{\sigma}^{(i)} \vec{r}_{ik})(\vec{\sigma}^{(k)} \wedge \vec{r}_{ik}) \left(\frac{1}{\varkappa r_{ik}} + \frac{1}{\varkappa^2 r_{ik}^2}\right)\right] \frac{e^{-\varkappa r_{ik}}}{4\pi r_{ik}}, \end{aligned}$$

where

$$\vec{x}^0 = \frac{|\vec{x}|}{r}.$$

The expression for Q is not given here as we will see that it does not play any rôle in these calculations. The expression (12) for the interaction operator is sufficiently accurate if the wavelength of the light quantum is small compared to the "radius" of the deuteron, a condition which is well fulfilled for the whole energy region of interest.

We now expand Ψ into the proper wave functions of the unperturbed problem:

$$\Psi = \Sigma a_n \Psi_n e^{-\frac{i}{h}(E_n + 2Mc^2)t} + \sum_{\mu} \int dE' \, a_{E'\mu} \Psi_{E'\mu} e^{-\frac{i}{h}(E' + 2Mc^2)t}. \quad (15)$$

Assuming the deuteron to be initially in the ground state (0), we have for $a_{E\mu}$, (and similarly for a_n but this quantity is of no interest here):

$$\left.\begin{aligned} a_{E\mu} = (E,\mu|\Omega|0)\frac{e^{-\frac{i}{h}(-E_0-E+h\nu)t} - 1}{-E_0 - E + h\nu} + (E,\mu|\Omega^*|0)\frac{e^{-\frac{i}{h}(-E_0-E-h\nu)t} - 1}{-E_0 - E - h\nu}, \\ (E,\mu|\Omega|0) = \int d\vec{x} \, \Psi_{E\mu}^* \Omega \Psi_0, \end{aligned}\right\} (16)$$

E_0 is the absolute value of the binding energy. The second term in the expression for $a_{E\mu}$ can be neglected as it will not give rise to resonance.

We now must obtain the asymptotic expression for (15); to find this we can neglect the contribution of the discrete spectrum and may write $\Psi_{E\mu}\,(r\to\infty)$ instead of $\Psi_{E\mu}$. Thus

$$\Psi_\infty = \sum_\mu \int dE'\,.(E',\mu\,|\,\Omega\,|\,0)\,e^{-\frac{i}{h}(E+2Mc^2)t}\,\Psi_{E'\mu}(r\to\infty)\,.\,\frac{e^{-\frac{i}{h}(E'-E)t}-1}{E'-E},$$

with

$$E = h\nu - E_0. \tag{17}$$

Expanding around the resonance value of E':

$$k' \cong k + \frac{dk}{dE}(E'-E),$$

$$(E',\mu\,|\,\Omega\,|\,0) \cong (E,\mu\,|\,\Omega\,|\,0),$$

and inserting (10), we get

$$\Psi_\infty = e^{-\frac{i}{h}(E+2Mc^2)t}\sum_\mu B_\xi(E,\mu;\vartheta,\varphi)\,.\,(E,\mu\,|\,\Omega\,|\,0)\,[e^{i\xi_j}J_1 \pm e^{-i\xi_j}J_2],$$

where the plus and minus sign hold for the large and small components respectively and

$$J_1 = \frac{e^{ikr}}{2r}\int_0^\infty dE'\,.\,\frac{e^{i\left(\frac{dk}{dE}r-\frac{t}{h}\right)(E'-E)} - e^{i\frac{dk}{dE}r(E'-E)}}{E'-E},$$

$$J_2 = \frac{e^{-ikr}}{2r}\int_0^\infty dE'\,.\,\frac{e^{-i\left(\frac{dk}{dE}r+\frac{t}{h}\right)(E'-E)} - e^{-i\frac{dk}{dE}r(E'-E)}}{E'-E}.$$

Now *)

$$\int_0^\infty \frac{e^{i\alpha(E'-E)} - e^{i\beta(E'-E)}}{E'-E}\,dE' \backsim \begin{cases} 0 \text{ if } \alpha \text{ and } \beta \text{ have the same sign,} \\ 2\pi i \text{ if } \alpha > 0 > \beta, \\ -2\pi i \text{ if } \alpha < 0 < \beta. \end{cases}$$

---

*) Cf. *loc. cit.* [11]), p. 446.

Thus, if we take $vt = \frac{dE}{dp} . t \rangle\rangle r$, so $\frac{dk}{dE} . r \langle\langle \frac{t}{h}$, we have

$$J_1 = -\pi i \frac{e^{ikr}}{r}, \qquad J_2 = 0$$

and

$$\Psi_\infty = -\pi i e^{-\frac{i}{h}(E+2Mc^2)t} \sum_\mu B_\xi(E, \mu; \vartheta, \Phi) . (E, \mu \mid \Omega \mid 0) \frac{e^{i(kr+\varepsilon_j)}}{r}.$$

With

$$\varrho\,(r \to \infty, \vartheta, \Phi) = \mid \Psi_\infty \mid^2 = \frac{\pi^2}{r^2} \mid \sum_\mu B_\xi . (E, \mu \mid \Omega \mid 0) e^{i\varepsilon_j} \mid^2, \qquad (18)$$

the differential cross section is, (for a fixed direction of polarization of the γ-rays)

$$d\Phi = \bar{\varrho}\, v r^2\, d\omega, \qquad (19)$$

where $\bar{\varrho}$ is meant to be the average over the three magnetic substates corresponding with the degeneracy of the ground state.

3. *Calculation of the cross sections* *). According to the prescription given in M.R. we must break off our calculations at the first stage which gives non-vanishing contributions to the effect concerned. We thus will not have to go further than to the first order in the velocities, that is to say, we will only have to consider matrix elements $(F \mid \Omega \mid 0)$ which belong to one of the following three types

$$\int \overset{(0)}{\Psi_F^*}\, \Omega\, \overset{(0)}{\Psi_0}\, dv, \qquad \int \overset{(1)}{\Psi_F^*}\, \Omega\, \overset{(0)}{\Psi_0}\, dv, \qquad \int \overset{(0)}{\Psi_F^*}\, \Omega\, \overset{(1)}{\Psi_0}\, dv.$$

It should be noted that, even if $(F \mid \Omega \mid 0)$ satisfies this requirement, it still may be negligible in our approximation if Ω itself contains velocity dependent factors.

a) *The PE effect.* We have to consider the transitions due to $\Omega_{el} = \vec{\mathfrak{E}}\, \vec{\mathfrak{P}}$. Taking the x-axis as direction of propagation of the photon beam, and the z-axis as the direction of its electric vector:

$$\Omega_{el} = A\, \mathfrak{P}_z. \qquad (20)$$

*) I should like to thank dr. MøLLER for the communication of preliminary calculations on the photo-effect which have provided a valuable check of the calculations given here.

We choose the amplitude A of the fields of the light wave such that it normalizes the radiation to one (polarized) photon per sec. per cm^2:

$$|A|^2 = \frac{h\nu}{2c}. \tag{21}$$

First we consider $\Omega_{\mathrm{el}}^{\mathrm{nucl}}$:

$$\Omega_{\mathrm{el}}^{\mathrm{nucl}} = \frac{e}{4} A \,(\tau_3^{(1)} - \tau_3^{(2)})\, z. \tag{22}$$

As the ground state is antisymmetric in the isotopic spins and

$$^3\zeta_0\,(\tau_3^{(1)} - \tau_3^{(2)})\,^1\zeta_0 = 2; \quad ^1\zeta_0\,(\tau_3^{(1)} - \tau_3^{(2)})\,^1\zeta^0 = 0,$$

the final state must be antisymmetric with respect to $\tau_3^{(i)}$. Taking further into account the behaviour of (22) with regard to rotations and spatial reflections and the fact that the ground state is of the type Ia with $l = 0$, $j = 1$, we find the following possibilities for the states that combine with the ground state, (behind each state we have indicated in brackets the spectroscopic symbol that corresponds to the non-relativistic approximation):

$$\begin{aligned} &\mathrm{I}\,a,\ l = 1, j = 2\ (^3\overline{P_2}), \\ &\mathrm{I}\,a,\ l = 1, j = 0\ (^3\overline{P_0}), \\ &\mathrm{I}\,b,\ l = j = 1 \quad (^3\overline{P_1}), \end{aligned}$$

while the familiar selection rule $\triangle m = 0$ holds. From (5) it is easily seen that to the first order in the velocities we have for all these transitions:

$$(F\,|\,\Omega_{\mathrm{el}}^{\mathrm{nucl}}\,|\,0) = \int \overset{(0)}{\Psi_F^*}\, \Omega_{\mathrm{el}}^{\mathrm{nucl}}\, \overset{(0)}{\Psi_0}\, dv. \tag{23}$$

Using (6) the matrix-elements can readily be calculated and we get

$$\left.\begin{aligned} &(\mathrm{I}\,a,\ l=1, j=2\,|\,\Omega_{\mathrm{el}}^{\mathrm{nucl}}\,|\,0) = -\frac{e}{2} A \,.\, I_1 \times \begin{cases} \sqrt{6}/6 & 1 \to 1 \\ \sqrt{2}/3 & 0 \to 0 \\ \sqrt{6}/6 & -1 \to -1 \end{cases} \\ &(\mathrm{I}\,a,\ l=1, j=0\,|\,\Omega_{\mathrm{el}}^{\mathrm{nucl}}\,|\,0) = \quad \frac{e}{6} A \,.\, I_1 \qquad\qquad\qquad 0 \to 0 \\ &(\mathrm{I}\,b,\ j=1 \qquad |\,\Omega_{\mathrm{el}}^{\mathrm{nucl}}\,|\,0) = \quad \frac{e}{2} A\, I_2 \times \begin{cases} \sqrt{6}/6 & 1 \to 1 \\ 0 & 0 \to 0 \\ -\sqrt{6}/6 & -1 \to -1, \end{cases} \end{aligned}\right\} \tag{24}$$

where we have indicated behind each expression the corresponding magnetic transition; further

$$I_1 = \int dr \,.\, r \,.\, R_a^*(1)\, R_0,$$

$$I_2 = \int dr \,.\, r\, R_b^*(0)\, R_0,$$

where R_0 is the large radial wave function of the ground state. Now, by (7):

$$R_a(1) = R_b(0) \equiv \bar{R} = \bar{R}^*, \tag{25}$$

so

$$I_1 = I_2 \equiv I = \int dr\, \bar{R}\, R_0\, r.$$

$\Omega_{\text{el}}^{\text{exch}}$, the second part of Ω_{el}, which is given by (see equ. (13)):

$$\Omega_{\text{el}}^{\text{exch}} = -\frac{e}{8\pi hc} A \,.\, \frac{g_1 g_2}{\varkappa} (\tau^{(1)} \wedge \tau^{(2)})_3 \{(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \wedge \vec{x}^0\}_z\, e^{-\varkappa r} \tag{26}$$

does, in our approximation, not contribute to the PE effect. This will be shown in the appendix.

With the help of (18) and (19) we can now directly obtain the differential cross section. As the final states all are P-states, the phase factors δ_j may be neglected in this case. We then get, with the help of (9) and (2) and neglecting those terms in B_ξ that are proportional to $(v/c)^2$, expressions which turn out to be the same for the three possible magnetic transitions, so that they directly give the average value $\bar{\varrho}$. The result is:

$$d\Phi^{\text{el}}(\vartheta) = \frac{e^2 \nu}{16 c} \left| \int_0^\infty \bar{R}\, R_0\, r\, dr \right|^2 \cos^2\vartheta \,.\, \sin\vartheta\, d\vartheta\, d\phi. \tag{27}$$

This is the cross section for a fixed direction of polarization. To obtain the cross section for an unpolarized photon, we have to average over all ϑ corresponding with the same Θ, (the angle between the direction of the light quantum and that of the neutron). This we do by first transforming to other angular variables:

$$\cos\vartheta = \sin\Theta \cos\psi, \quad \sin\vartheta\, d\vartheta\, d\phi = \sin\Theta\, d\Theta\, d\psi,$$

where ψ is related to Θ in the same way as Φ to ϑ; we then integrate over all orientations ψ of the polarization direction (which gives us a "zonal cross section" for an unpolarized photon), and multiply the result by $d\psi/2\pi$. This gives, on account of the factor $\cos^2\psi$, an averaging factor ½ in the differential cross section:

$$d\Phi^{\mathrm{el}}(\Theta) = \frac{e^2 \nu}{2 . 16\, c} \left| \int_0^\infty dr\, \overline{R}\, R_0\, r \right|^2 \sin^2\Theta \, . \, \sin\Theta\, d\Theta\, d\psi. \tag{28}$$

The total cross section is

$$\Phi^{\mathrm{el}} = \frac{\pi e^2 \nu}{12 c} \left| \int_0^\infty dr\, \overline{R}\, R_0\, r \right|^2 . \tag{29}$$

This result is identical in form with that obtained in the BETHE–PEIERLS theory. Deviations from this simple formula are at most to be expected in the second order with respect to the velocities.

b) The PM effect. To begin with we consider the contribution of the first term in (14), for which we write, noting that the magnetic vector stands in the y-direction

$$\Omega^{\mathrm{nucl}}_{\mathrm{magn}} = \frac{e}{8} A \, [\vec{x} \wedge \{(1-\tau_3^{(1)})\, \vec{\alpha}^{(1)} - (1-\tau_3^{(2)})\, \vec{\alpha}^{(2)}\}]_y \, .$$

Now

$${}^3\zeta_0\,(1-\tau_3^{(1)})\,{}^1\zeta_0 = -1 \quad , \quad {}^3\zeta_0\,(1-\tau_3^{(2)})\,{}^1\zeta_0 = 1$$
$${}^1\zeta_0\,(1-\tau_3^{(1)})\,{}^1\zeta_0 = 1, \quad , \quad {}^1\zeta_0\,(1-\tau_3^{(2)})\,{}^1\zeta_0 = 1,$$

so there are allowed transitions to states which are symmetric as well as antisymmetric in the isotopic spins. In the second case however (transition to a^3D-state) the contributions are vanishingly small in the whole energy-region which is of interest; we neglect them in the following. If the final state is symmetric in the τ's we can write

$$\Omega^{\mathrm{nucl}}_{\mathrm{magn}} = -\frac{e}{8} A \, \{\vec{x} \wedge (\vec{\alpha}^{(1)} + \vec{\alpha}^{(2)})\}_y ;$$

the only allowed transitions are to the state of the type IIb with

$j = 0$ (^1S), while $\triangle m = \pm 1$. With the help of (6) we get (for $1 \to 0$ as well as $-1 \to 0$)

$$(\mathrm{II}b, j = 0 \mid \Omega^{\mathrm{nucl}}_{\mathrm{magn}} \mid 0) = i\,\mu_0\, A\, \frac{\sqrt{2}}{12} \int_0^\infty dr\, r \left[R_0 \left(\frac{d}{dr} - \frac{1}{r} \right) R_{II}(0) + \right.$$
$$\left. + R_{II}(0) \left(\frac{d}{dr} - \frac{1}{r} \right) R_0 \right) \Big],$$

where $\mu_0 = eh/2Mc$ is the nuclear magneton. The integral can be simplified by partial integrations. The result is

$$(\mathrm{II}b, j = 0 \mid \Omega^{\mathrm{nucl}}_{\mathrm{magn}} \mid 0) = -i\,\mu_0\, A\, \frac{\sqrt{2}}{4} \int_0^\infty dr\, R\, R_0 \quad ; \quad R = R_{II}(0). \quad (30)$$

The extra magnetic moments of proton and neutron are contained in the second part of (22), namely those terms for which $i = k$ ($= 1,2$)*). These terms are "of course" infinite and can only be managed by using a cut-off prescription. Calling their contribution to (19) $\Omega^{\mathrm{extra}}_{\mathrm{magn}}$, we have

$$\Omega^{\mathrm{extra}}_{\mathrm{magn}} = \lim_{\varrho = 0} -\frac{g_2^2}{4\pi h c}\, i e A \sum_{i=1,2} \tau_3^{(i)} \left[2 i \vec{\sigma}^{(i)} \left(\frac{1}{\varkappa^2 \varrho} - \frac{1}{2\varkappa} \right) + \right.$$
$$\left. + (\vec{\sigma}^{(i)} \vec{\varrho})\, (\vec{\sigma}^{(i)} \wedge \vec{\varrho}) \left(\frac{1}{\varkappa^2 \varrho} + \frac{1}{\varkappa} \right) \right]_y e^{-\varkappa\varrho}.$$

Averaging over all directions of $\vec{\varrho}$ (which we indicate by the overlining of the left member) gives for any compononent $(\vec{\varrho}\,\vec{e})$ of $\vec{\varrho}$

$$\overline{(\vec{\sigma}^{(i)} \vec{\varrho})\, (\vec{\varrho}\, \vec{e})} = \tfrac{1}{3}\, (\vec{\sigma}^{(i)} \vec{e}).$$

Thus if we take for $(\vec{\sigma}^{(i)} \vec{\varrho})\, (\vec{\sigma}^{(i)} \wedge \vec{\varrho})_y$ its average over all directions of $\vec{\varrho}$, which seems to be an appropriate way to deal with this quantity, we have to replace it by $^1/_3\, (\sigma_x^{(i)} \sigma_z^{(i)} - \sigma_z^{(i)} \sigma_x^{(i)}) = -2i/3\, \sigma_y^{(i)}$.

*) Other terms of higher order, which also contribute to these extra magnetic moments, have to be discarded according to the prescription given by MøLLER and ROSENFELD [12]).

We now cut off by replacing

$$\lim_{\varrho=0} \frac{g_2^2}{4\pi hc} \cdot \frac{M}{M_m} \cdot \frac{1}{3} \left(\frac{4}{\varkappa\varrho} - 5 \right) e^{-\varkappa\varrho}$$

by a finite quantity which we call μ and obtain

$$\Omega_{\text{magn}}^{\text{extra}} = -\mu_0 \, \mu \, A \sum_i \tau_3^{(i)} \sigma_y^{(i)}.$$

For this operator only the ^1S-state is allowed as final state. The matrix-elements are of the type as indicated in (23). The result is

$$(\text{II}\,b,\ j=0 \,|\, \Omega_{\text{magn}}^{\text{extra}} \,|\, 0) = -i\,\mu_0\,\mu\,A\sqrt{2}\int_0^\infty dr\,R R_0.$$

We notice that

$$2\mu(\text{II}b,\ j=0 \,|\, \Omega_{\text{magn}}^{\text{nucl}} \,|\, 0) = \tfrac{1}{2}\ (\text{II}b,\ j=0 \,|\, \Omega_{\text{magn}}^{\text{extra}} \,|\, 0),$$

the perhaps unexpected factor $\frac{1}{2}$ on the right arising from the fact that we have to do with the magnetic moment of the deuteron with respect to its centre of gravity [12].

At the moment the only way of dealing with μ is to fix it with the empirical values for μ_P and μ_N (the magnetic moments of proton and neutron in units μ_0):

$$\mu_P = 1 + \mu;\ \ \mu_N = -\mu. \tag{31}$$

This gives

$$(\text{II}b,\ j=0 \,|\, \Omega_{\text{magn}}^{\text{nucl}} + \Omega_{\text{magn}}^{\text{extra}} \,|\, 0) = -i\mu_0 A \frac{\sqrt{2}}{2}(\mu_P - \mu_N - \tfrac{1}{2}) \int_0^\infty dr\,R R_0. \tag{32}$$

Finally we have to consider the terms from the second part of (14) with $i \neq k$; we call the corresponding operator $\Omega_{\text{magn}}^{\text{exch}}$ which reduces after some simple calculations to

$$\Omega_{\text{magn}}^{\text{exch}} = eA \cdot \frac{g_2^2}{4\pi hc} (\tau^{(1)} \wedge \tau^{(2)})_3 \left[(\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)})_y \left(\frac{1}{\varkappa^2 r} - \frac{1}{2\varkappa} \right) - \right.$$
$$\left. - \tfrac{1}{2} \{ (\vec{\sigma}^{(1)} \wedge \vec{x}^0)_y (\vec{\sigma}^{(2)} \vec{x}^0) - (\vec{\sigma}^{(2)} \wedge \vec{x}^0)_y (\vec{\sigma}^{(1)} \vec{x}^0) \} \cdot \left(\frac{1}{\varkappa^2 r} + \frac{1}{\varkappa} \right) \right] e^{-\varkappa r}.$$

As ${}^3\zeta_0 (\boldsymbol{\tau}^{(1)} \wedge \boldsymbol{\tau}^{(2)})_3 \; {}^1\zeta_0 = 2i$, ${}^1\zeta_0 (\boldsymbol{\tau}^{(1)} \wedge \boldsymbol{\tau}^{(2)})_3 \; {}^1\zeta_0 = 0$, the final states must be antisymmetric in the isotopic spins. The only states that combine with the ground state turn out to be

Type II b, $j = 0$, (${}^1S$),

Type II b, $j = 2$, (${}^1D$).

The matrix-elements are found to be again of the type (23). They are

$$\left.\begin{matrix} 1 \to 0 \\ -1 \to 0 \end{matrix}\right\} (\mathrm{II}\, b,\ j = 0 \,|\, \Omega^{\mathrm{exch}}_{\mathrm{magn}} \,|\, 0) = $$

$$= \frac{g_2^2}{4\pi hc}\, i \mu_0 \frac{M}{M_m} A \frac{2\sqrt{2}}{3} \int_0^\infty dr\, R\, R_0 \left(\frac{4}{\varkappa r} - 5 \right) e^{-\varkappa r}, \tag{33}$$

$$\left.\begin{matrix} 1 \to 0 \\ -1 \to 0 \end{matrix}\right\} (\mathrm{II}\, b,\ j = 2 \,|\, \Omega^{\mathrm{exch}}_{\mathrm{magn}} \,|\, 0) = $$

$$= - \frac{g_2^2}{4\pi hc}\, i \mu_0 \frac{M}{M_m} A \frac{2\sqrt{10}}{15} \int_0^\infty dr\, R_{II}(2)\, R_0 \left(\frac{1}{\varkappa r} + 1 \right) e^{-\varkappa r}. \tag{34}$$

We have verified that inclusion of the latter modifies only very slightly the final results and we will therefore ignore its contribution. The transitions to D-states, to which the electric quadrupole moment gives rise are also small and will be neglected.

The differential cross-section is computed from (32) and (33) in the same way as (28) was found, (averaging over the directions of polarization here gives a factor 2π). The result is

$$d\,\Phi^{\mathrm{magn}} = \tfrac{1}{3}\, \mu_0^2 \frac{v}{c} \, |J|^2 \, . \sin \Theta \, d\Theta \, d\psi \tag{35}$$

with

$$J = - \frac{\sqrt{2}}{2} (\mu_P - \mu_N - \tfrac{1}{2}) \int_0^\infty dr\, R\, R_0 + $$

$$+ \frac{g_2^2}{4\pi hc} \frac{M}{M_m} \frac{2\sqrt{2}}{3} \int_0^\infty dr\; R\, R_0 \left(\frac{4}{\varkappa r} - 5 \right) e^{-\varkappa r} dr. \tag{36}$$

Therefore

$$\Phi^{\text{magn}} = \frac{4\pi}{3}\,\mu_0^2 \cdot \frac{\nu}{c}\,|J|^2. \tag{37}$$

c) Numerical estimations. For the numerical evaluation of the cross-sections we have taken for the large radial wave function of the ground state the approximate expression obtained by WILSON under the assumption of a nuclear potential of the type as used here:

$$R_0(r) = \sqrt{\frac{\alpha^3 \varkappa^3}{2}}\, e^{\frac{-\alpha \varkappa r}{2}}\, r. \tag{38}$$

Further we take $E_0 = 2{,}16$ MeV. Assuming

$$\frac{M_m}{M} = \frac{1}{10},$$

we have $\alpha = 2{,}13$ and (cf. M.R. (107), (108) (109)):

$$\frac{g_1^2}{4\pi\, hc} = 0.027, \quad \frac{g_2^2}{4\pi\, hc} = 0.065.$$

For the phase δ_0 of the ^1S state which occurs as final state in the magnetic transitions we have, according to BETHE and BACHER [10]),

$$\cos\delta_0 = -\frac{\beta}{\sqrt{\beta^2 + k^2}}, \qquad \sin\delta_0 = \frac{k}{\sqrt{\beta^2 + k^2}}, \tag{39}$$

where

$$\beta = \pm \frac{\sqrt{\mathrm{M}E_0'}}{h}. \tag{40}$$

The plus or minus sign holds according to whether the first excited (^1S) state of the deuteron (with the energy E_0') is real or virtual. Experiments have decided for the latter possibility. We have taken $E_0' = 105.000$ eV. Further we put *)

$$\overline{R} = \lambda \sqrt{\frac{2}{\pi}} \left(-\cos kr + \frac{\sin kr}{kr} \right), \tag{41}$$

$$R = \lambda \sqrt{\frac{2}{\pi}}\, \sin(kr + \delta_0).$$

*) See *loc. cit.* [10]), p. 124 equ. (77*b*), (77*c*) and p. 128 equ. (93*a*).

Inserting all this in (29) and (36) and carrying out all integrations we get *)

$$\left.\begin{aligned} d\Phi^{\mathrm{el}}(\Theta) = \frac{\alpha^5}{2}\cdot\frac{e^2}{hc}\cdot\left(\frac{Mc}{h}\right)^2\cdot\frac{h\nu}{Mc^2}\cdot\left(\frac{\varkappa}{k}\right)^5\cdot\frac{1}{k^4}\cdot \\ \cdot\left(\frac{1}{a(k)}\right)^6 \sin^2\Theta\,.\,\sin\Theta\, d\Theta\,.\,\frac{d\psi}{2\pi} \end{aligned}\right\} \tag{42a}$$

$$\Phi^{\mathrm{el}} = \frac{2\alpha^5}{3}\cdot\frac{e^2}{hc}\cdot\left(\frac{Mc}{h}\right)^2\cdot\frac{h\nu}{Mc^2}\left(\frac{\varkappa}{k}\right)^5\cdot\frac{1}{k^4}\cdot\left(\frac{1}{a(k)}\right)^6, \tag{42b}$$

$$d\Phi^{\mathrm{magn}} = \frac{\alpha^3}{12}\cdot\frac{e^2}{hc}\cdot\frac{h\nu}{Mc^2}\left(\frac{\varkappa}{k}\right)^3\cdot\frac{1}{k^2+\beta^2}\,|B^{\mathrm{magn}}|^2\,.\sin\Theta\, d\Theta\,.\,\frac{d\psi}{2\pi}, \tag{43a}$$

$$\Phi^{\mathrm{magn}} = \frac{\alpha^3}{6}\cdot\frac{e^2}{hc}\cdot\frac{h\nu}{Mc^2}\left(\frac{\varkappa}{k}\right)^3\cdot\frac{1}{k^2+\beta^2}\,|B^{\mathrm{magn}}|^2, \tag{43b}$$

with

$$\left.\begin{aligned} B^{\mathrm{magn}} = -(\mu_P-\mu_N-\tfrac{1}{2})\frac{\sqrt{2}}{2}\left[-\alpha\frac{\beta\varkappa}{k^2}+\alpha^2\left(\frac{\varkappa}{k}\right)^2-1\right]\cdot\left(\frac{1}{a(k)}\right)^2 + \\ + \frac{g_2^2}{4\pi hc}\cdot\frac{M}{M_m}\cdot\frac{2\sqrt{2}}{3}\left[-\frac{\beta}{\varkappa}\left\{\frac{4}{b(k)}-5(\alpha+2)\left(\frac{\varkappa}{k}\right)^2\cdot\left(\frac{1}{b(k)}\right)^2\right\} + \right. \\ \left. + \frac{2(\alpha+2)}{b(k)} - 5\left\{\frac{(\alpha+2)^2}{4}\left(\frac{\varkappa}{k}\right)^2-1\right\}\left(\frac{1}{b(k)}\right)^2\right]; \end{aligned}\right\} \tag{44}$$

$$a(k) = 1+\frac{\alpha^2}{4}\left(\frac{\varkappa}{k}\right)^2, \qquad b(k) = 1+\frac{(\alpha+2)^2}{4}\left(\frac{\varkappa}{k}\right)^2. \tag{45}$$

We notice that for very large energies both the PE and the PM cross sections decrease like $\nu^{-7/2}$, that is more rapidly than in the "old" theory ($\backsim \nu^{-3/2}$). Apart from that we may state in a general way that the results for not too large frequencies are of the same order of magnitude as in the old theory. As regards the absolute values of the cross sections we have thus reasonable agreement with the measured values, *viz.* 5.10^{-28} cm^2 (CHADWICK and GOLDHABER [1])) or 9.10^{-28} cm^2 (VON HALBAN [6])) for the γ-rays

*) It should be borne in mind that e is expressed in Heaviside units in accordance with the normalization (21) of A.

of ThC'' ($h\nu = 2{,}64$ MeV), and $11{,}6 \,.\, 10^{-28}$ for an energy of 6,2 MeV (ALLEN and SMITH [13])).

As regards the angular distribution, it seems that the terms specific for meson theory, (arising from $\Omega^{\text{exch}}_{\text{magn}}$), do not appreciably ameliorate the situation. However, we are unable as yet to decide this point, since we have noticed that the wave functions at our disposal are much too unreliable to allow definite statements regarding such a sensitive effect (see also the Appendix).

5. *Capture of neutrons by protons.* The cross-sections for these processes can immediately be inferred from (42) and (43). We have in fact, calling the cross-sections for "electric" and "magnetic" capture Φ^{el}_c and Φ^{magn}_c respectively

$$\Phi^{\text{el}}_c = \frac{9}{2}\left(\frac{\nu}{kc}\right)^2 \Phi^{\text{el}};\ \Phi^{\text{magn}}_c = \frac{3}{2}\left(\frac{\nu}{kc}\right)^2 \Phi^{\text{magn}}.$$

Therefore

$$\Phi^{\text{el}}_c = 3a^5 \,.\, \frac{e^2}{hc} \cdot \left(\frac{Mc}{h}\right)^2 \cdot \frac{h\nu}{Mc^2} \cdot \left(\frac{\varkappa}{k}\right)^5 \cdot \frac{1}{k^6} \cdot \left(\frac{1}{a(k)}\right)^6, \quad (46)$$

$$\Phi^{\text{magn}}_c = \frac{\alpha^3}{4} \cdot \frac{e^2}{hc} \cdot \frac{h}{Mc} \cdot \left(\frac{\nu}{c}\right)^3 \left(\frac{\varkappa}{k}\right)^3 \cdot \frac{1}{k^2(k^2+\beta^2)} |B^{\text{magn}}|^2. \quad (47)$$

We are especially interested in the behaviour of these expressions in the region of thermal neutron energies. In this region we may write β^2 for $k^2 + \beta^2$ (as $\beta^2 = 25{,}7 \,.\, 10^{22}$ cm^{-2}).

It is then easily seen that, for thermal neutrons, $\Phi^{\text{el}}_c \backsim k$, while $\Phi^{\text{magn}}_c \backsim k^{-1}$. Φ^{el}_c can thus be ignored, while, just as in the "old" theory Φ^{magn}_c gives us the well known $1/v$ law. The agreement with the experimentally found values: $0{,}27 \,.\, 10^{-24}$ cm^2 for a velocity of $2{,}2 \,.\, 10^5$ cm. sec.$^{-1}$ (FRISCH, V. HALBAN and KOCH [14])) and $0{,}31 \,.\, 10^{-24}$ cm^2 for a velocity of $2{,}5 \,.\, 10^5$ cm. sec.$^{-1}$ which was obtained by AMALDI and FERMI [15]) appears to be satisfactory.

APPENDIX.

To calculate the PE effect we have made use of operator $\vec{\mathfrak{E}}\vec{\mathfrak{P}}$. But, as

$$\vec{\mathfrak{A}}\dot{\vec{\mathfrak{P}}} = \vec{\mathfrak{E}}\vec{\mathfrak{P}} + \frac{d}{c\,dt}(\vec{\mathfrak{A}}\vec{\mathfrak{P}}),$$

and as the second term on the right has vanishing matrix elements for the transitions concerned (because of energy conservation), we may use the operator $\vec{\mathfrak{A}}\,\dot{\vec{\mathfrak{P}}}$ just as well. This we will do here; in the centre of gravity system we have

$$\dot{\vec{\mathfrak{P}}} = \frac{e}{2}\sum_{1,2}(1-\tau_3^{(i)})\,\vec{\alpha}^{(i)} + \frac{e}{4\pi hc}(\tau^{(1)}\wedge\tau^{(2)})_3\,\{g_1^2+g_2^2\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\}\,\vec{x}^0\,e^{-\varkappa r}. \quad (48)$$

Denoting $\vec{\mathfrak{A}}\,\dot{\vec{\mathfrak{P}}}$ by $\overline{\Omega}_{\mathrm{el}}$ we have for the first part of this operator:

$$\overline{\Omega}_{\mathrm{el}}^1 = \frac{A'e}{2}\,[(1-\tau_3^{(1)})\,\alpha_z^{(1)} + (1-\tau_3^{(2)})\,\alpha_z^{(2)}],$$

($A' = ic/\nu\, A$ is the amplitude of the vector potential). This gives rise to the following matrix elements:

$$(\mathrm{I}\,a,\ l=1, j=2\mid\overline{\Omega}_{\mathrm{el}}^1\mid 0) = -e A\, I_\alpha \quad \begin{cases} 1 & 1\to 1 \\ {}^2/_3\sqrt{3} & 0\to 0 \\ 1 & -1\to -1 \end{cases}$$

$$(\mathrm{I}\,a,\ l=1, j=0\mid\overline{\Omega}_{\mathrm{el}}^1\mid 0) = \frac{1}{2}A'e\, I_\beta \qquad\qquad 0\to 0$$

$$(\mathrm{I}\,b,\ j=1\mid\overline{\Omega}_{\mathrm{el}}^1\mid 0) = \frac{\sqrt{2}}{4}A'e\, I_\gamma \quad \begin{cases} 1 & 1\to 1 \\ 0 & 0\to 0 \\ -1 & -1\to -1 \end{cases}$$

With

$$I_\alpha = \int_0^\infty C_0\,(j=1)\,R_a^*\,(1),$$

$$I_\beta = -\tfrac{1}{3}\sqrt{6}\int_0^\infty C_0\,(j=1)\,R_a^*\,(1) + \int_0^\infty C_2^*\,(j=0)\,R_0,$$

$$I_\gamma = \int_0^\infty C_1^*\,(j=1)\,R_b\,(0).$$

Now (see (25)) $R_a\,(1) = R_b\,(0) = \overline{R}$. Taking into account the equations (11), we get introducing

$$I_I = \int_0^\infty \overline{R}\left(\frac{d}{dr}-\frac{1}{r}\right)R_0\,dr,$$

$$\left.\begin{aligned}
(\mathrm{I}\,a,\, l=1, j=2 \mid \overline{\Omega}^1_{\mathrm{el}} \mid 0) &= -\frac{ieA'h}{Mc} I_I \begin{cases} \sqrt{6}/6 \\ \sqrt{2}/3 \\ \sqrt{6}/6 \end{cases} \\
(\mathrm{I}\,a,\, l=1, j=0 \mid \overline{\Omega}^1_{\mathrm{el}} \mid 0) &= -\frac{ieA'h}{Mc} I_I \begin{cases} 0 \\ -1/3 \\ 0 \end{cases} \\
(\mathrm{I}\,b,\, j=1 \mid \overline{\Omega}^1_{\mathrm{el}} \mid 0) &= -\frac{ieA'h}{Mc} I_I \begin{cases} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \end{cases}
\end{aligned}\right\} \quad (49)$$

The second part of (48), (the corresponding operator is called $\overline{\Omega}^2_{\mathrm{el}}$), must be treated in the same way. We obtain expressions entirely similar to (49); we must in (49) only replace

$$\frac{h}{Mc} I_I \text{ by } -\frac{g_1^2 + g_2^2}{2\pi hc} I_{II},$$

where

$$I_{II} = \int_0^\infty dr\, e^{-\varkappa r}\, \overline{R}\, R_0.$$

Now it is easy to see that

$$\left.\begin{aligned}
\frac{\nu}{2c}\int dr\, \overline{R}\, R_0\, r = -\frac{h}{Mc}\int dr\, \overline{R}\left(\frac{d}{dr} - \frac{1}{r}\right) R_0 + \\
+ \frac{g_1^2 + g_2^2}{2\pi hc}\int dr\, e^{-\varkappa r}\, \overline{R}\, R_0,
\end{aligned}\right\} \quad (50)$$

In fact, $\overline{R}$ and R_0 satisfy the following equations:

$$\left[\frac{h^2}{M}\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) + E - \frac{g_1^2 + g_2^2}{4\pi} \cdot \frac{e^{-\varkappa r}}{r}\right] \overline{R} = 0,$$

$$\left[\frac{h^2}{M} \qquad - E_0 + \frac{3\,(g_1^2 + g_2^2)}{4\pi} \cdot \frac{e^{-\varkappa r}}{r}\right] R_0 = 0.$$

Subtracting these equations ($E + E_0 = h\nu$!) and multiplying the result by $r/2$, (50) readily follows.

Therefore

$$(F \,|\, \bar{\Omega}_{el} \,|\, 0) = (F \,|\, \vec{\mathcal{E}}\, \vec{\mathcal{P}}_{nucl} \,|\, 0).$$

That the matrixelements of $(F \,|\, \vec{\mathcal{E}}\, \vec{\mathcal{P}}_{exch} \,|\, 0)$ indeed vanish in our approximation can also directly be seen. For these are all proportional to

$$\frac{\nu}{\varkappa c} \cdot \frac{g_1 g_2}{4\pi hc} \int dr\, e^{-\varkappa r}\, \bar{R}\, R_0,$$

and thus are obviously of higher order in the velocities compared to those of $\bar{\Omega}^2_{el}$. Now the matrixelements of $\bar{\Omega}^2_{el}$ and $\bar{\Omega}^1_{el}$ are of the same order, as follows from (52) below, which proves our assumption.

Using the right member of (50) to calculate the PE cross section, we find, in the same way as we have derived (42),

$$d\,\Phi^{el}(\Theta) = \frac{\alpha^3}{8} \cdot \frac{e^2}{hc} \cdot \frac{Mc^2}{h\nu} \cdot \left(\frac{\varkappa}{k}\right)^3 \cdot \frac{1}{k^2} \cdot |B^{el}|^2 \,.\, sin^2\,\Theta \,.\, sin\,\Theta\, d\,\Theta\, \frac{d\psi}{2\pi}, \quad (51a)$$

$$\Phi^{el} = \frac{\alpha^3}{6} \cdot \frac{e^2}{hc} \cdot \frac{Mc^2}{h\nu} \cdot \left(\frac{\varkappa}{k}\right)^3 \cdot \frac{1}{k^2} \cdot |B^{el}|^2, \quad (51b)$$

with

$$B^{el} = \alpha\, \frac{M_m}{M} \cdot \left(\frac{1}{a(k)}\right)^2 + \frac{g_1^2 + g_2^2}{4\pi\, hc} \cdot \left(\frac{2}{b(k)}\right)^2. \quad (52)$$

On account of (50), (51) and (42) are mathematically equivalent. If one inserts numerical values in these equations, however, one meets with a serious discrepancy between the final results so obtained which clearly is a consequence of the approximative character of the wave functions we have used.

Finally we remark that the first term on the right of (52), arising from $\bar{\Omega}^1_{el}$ is of the same order in the velocities as the second which arises from $\bar{\Omega}^2_{el}$. This completes our proof that the matrix-elements of $\vec{\mathcal{E}}\, \vec{\mathcal{P}}_{exch}$ vanish in this approximation.

It should be noted that this result is also valid in a pure vector meson theory provided the dipole interaction potential (including cut-off) may be regarded as a perturbation.

REFERENCES.

1) J. CHADWICK and M. GOLDHABER, Nature, **134**, 237, 1934.
2) H. A. BETHE and R. PEIERLS, Proc. Roy. Soc. A **148**, 146, 1935.
3) H. MASSEY and C. MOHR, Proc. Roy. Soc. A **148**, 206, 1935.
4) E. FERMI, Phys. Rev. **48**, 570, 1935.
5) G. BREIT and E. U. CONDON, Phys. Rev. **49**, 904, 1936.
6) H. V. HALBAN, Nature **141**, 644, 1938.
7) C. MøLLER and L. ROSENFELD, D. Danske Vid. Selsk. math.-fys. Medd., **17**, fasc. 8, 1940.
8) H. FRÖHLICH, W. HEITLER and B. KAHN, Proc. Roy. Soc. A **174**, 85, 1940.
9) N. KEMMER, Helv. Phys. Acta, **10**, 47, 1937.
10) H. A. BETHE and R. F. BACHER, Rev. Mod. Phys. **8**, 82, 1936.
11) H. A. BETHE, Ann. d. Phys. **4**, 443, 1930.
12) C. MøLLER and L. ROSENFELD, *in course of publication.*
13) J. A. ALLEN and N. M. SMITH, Phys. Rev. **59**, 618, 1941.
14) O. R. FRISCH, H. V. HALBAN and J. KOCH, D. Danske Vid. Selsk. math.-fys. Medd., **15**, fasc. 10, 1937.
15) E. AMALDI and E. FERMI, Phys. Rev. **50**, 899, 1936.

CONSPECTUS.

In hujus dissertationis primo capite explicatur theoria jam a compluribus auctoribus elaborata, qua campus gravificus et campus electromagneticus in spatio projectivo quintidimensionali complecti possunt; deinde ducitur expressio generalis tensoris densitatis energiae et momenti cujuslibet campi. Eum tensorem legum campis impositarum causa symmetricum esse et evanescentem divergentiam habere ostenditur, qua secundo loco dicta proprietate energiae momenti ac electricitatis conservatio exprimitur. In quarta paragrapho adhibetur formula generalis ad calculationem tensoris energiae et momenti campi Diraciani.

In capite secundo extensione theoriae projectivae ad campos mesicos ostenditur, quomodo in illa theoria Møller–Rosenfeldiana de viribus nuclearibus incorporanda sit. Hac tractatione etiam minuitur numerus universalium constantium quae camporum mesicorum intensitatem determinant, et naturaliter introducitur interactio mesonum cum campo electromagnetico. Postquam deinde ostensum est, quomodo campus electronico-neutrinicus tractandus sit, expressione tensoris energiae ac momenti constituta, systematis functio Hamiltoniana et electricitatis distributio ducitur. Regulae commutationis variabilium campos mesicos describientium in compendiosiorem formam traducuntur. Functio Hamiltoniana postea transformatur separanda longitudinali parte campi electrici et statica parte campi mesici. Ejusdem methodi applicatio ad electricitatis distributionem transformandam accuratius discutitur et expressiones indicantur momentorum dipoli et quadrupoli electrici necnon dipoli magnetici nuclearis systematis.

In tertio capite photodisintegratio deuteronis et captatio neutronum a protonibus ex theoria Møller–Rosenfeldiana de viribus nuclearibus tractantur. Expressionem generalem sectionis efficaciae effectus photoelectrici invenitur formaliter identicam esse ei, quae ex anteriore Bethe–Peierlsiana theoria sequitur, sectionem efficaciam photomagneticam autem insuper accessionem ex mesico campo orientem continere. Sectiones efficaciae energia incidentis photonis crescente celerius minuuntur quam anterioris theoriae ratione. Magnitudines absolutae sectionum efficaciarum ejusdem magnitudinis categoriae sunt, atque magnitudines empirice inventae, quamquam certi numeris expressi effectus nondum dari possunt propter incertas functiones undarum deuteronis quibus usi sumus. Haec res certum judicium de angulari distributione praematurum reddit. Sectiones efficaciae captationis eaedem veram magnitudinis categoriam habent et, sicut in anteriore theoria, legem $1/v$ magneticum effectum esse apparet.

STELLINGEN

I

Wil men de mesontheorie zodanig formuleren, dat deze covariant is voor een continue 5-dimensionale groep van transformaties, dan is van fysisch standpunt de projectieve formulering te verkiezen.

II

In een systeem, bestaande uit een combinatie van een vectoriëel en een pseudoscalair mesonveld kan men een rechtstreekse wisselwerking tussen deze velden invoeren, door aan de Lagrange-functie extra termen toe te voegen. Eist men de in de vorige stelling genoemde invariantie, dan legt dit aan deze termen een grotere beperking op dan de gebruikelijke eis van invariantie t.o.v. de Lorentz-groep doet.

III

Metingen van de warmte, die vrijkomt bij een Uraan-splijting zijn niet te interpreteren zonder een gedetailleerde kennis van alle secundaire processen, die op de splijting volgen.

Vgl. M. C. HENDERSON, Phys. Rev. **58**, 774, 1940.

IV

Het is mogelijk, een differentiaalvergelijking op te stellen, die aangeeft hoe een groot-kanoniek ensemble ontstaat.

V

Het vereenvoudigde model van FURRY voor het kaskade-shower fenomeen kan zodanig verfijnd worden, dat het meer in overeenstemming is met het werkelijke proces. Het aldus verkregen model heeft qualitatief dezelfde eigenschappen als het FURRY-model.

H. FURRY, Phys. Rev. **52**, 569, 1937.

VI

De berekeningen van HAFSTAD en TELLER tonen aan, dat men uit de gegevens omtrent de bindingsenergieën en spectra van lichte atoomkernen niets met zekerheid kan zeggen over de bouw van die kernen.

L. R. HAFSTAD en E. TELLER, Phys. Rev. **54**, 681, 1938.

VII

Het is gewenst de werkzame doorsneden voor de foto-desintegratie van het deuteron te meten voor zeer hoge foton-energieën.

VIII

DIRAC's opvattingen betreffende een „mathematical quality in nature" is onaanvaardbaar.

P. A. M. DIRAC, Proc. Edinburgh Soc. **59**, 122, 1939.

IX

Men kan de definitie van „pool van een punt t.o.v. een puntenpaar op een rechte" uitbreiden tot „poolfiguur van een p-dimensionale vlakke ruimte t.o.v. een lineair k-stelsel hyperkwadrieken in een n-dimensionale ruimte", $(p \leqq n)$. Het aantal dimensies van deze laatste figuur is $n - |k - p| - 1$, $(n - 1 \geqq |k - p|)$, zijn graad is $\frac{(k+1)!}{(k-p+1)!\,p!}$, (als $p \leqq k$), $\frac{(p+1)!}{(p-k+1)!\,k!}$, (als $p \geqq k$).

X

Het is zeer gewenst, dat voor candidaten in de natuurkunde hetzij een college over „algemene methoden in de fysica" gegeven wordt, hetzij in seminaria of colloquia aandacht aan dit onderwerp wordt besteed.

XI

Het verdient aanbeveling om in wiskunde-schoolboeken de uitdrukking „bewijs uit het ongerijmde" te vervangen door „bewijs van de ongerijmdheid van het tegendeel", (zo men deze methode al wenst te handhaven).