# A contribution to the theory of eclipsing binaries

# A CONTRIBUTION TO THE THEORY

OF

# ECLIPSING BINARIES

## PROEFSCHRIFT

TER VERKRIJGING VAN DEN GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DEN RECTOR-MAGNIFICUS MR. J. C. NABER, HOOGLEERAAR IN DE FACULTEIT DER RECHTSGELEERDHEID, VOLGENS BESLUIT VAN DEN SENAAT DER UNIVER[illegible]TEIT TEGEN DE BEDENKINGEN VAN DE FACU[illegible]EIT DER WIS- EN NATUURKUNDE TE VER[illegible] OP DINSDAG 27 MAART 1923, DES NAM[illegible] 4 UUR, DOOR

# RECHERCHES ASTRONOMIQUES

## DE L'OBSERVATOIRE

## D'UTRECHT

## IX

# RECHERCHES ASTRONOMIQUES

## DE L'OBSERVATOIRE D'UTRECHT

# IX

PART I



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# A CONTRIBUTION TO THE THEORY

OF

# ECLIPSING BINARIES

BY

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# INTRODUCTION.

During the past 18 years Prof. A. A. NIJLAND, Director of the Utrecht Observatory, has observed a great number of variable stars, among which some 70 binaries of the *Algol* (and β *Lyrae*)-group. The determination of the light-curves of these binaries has been examined and, in trying to derive the best elements for some of these systems, I struck upon a modification of RUSSELL'S well-known method. The revised method has been applied to the light-curves of four stars, with a view of extending this work to the remaining stars in subsequent publications.

I am greatly indebted to Professor NIJLAND for his readiness in putting the material necessary for this investigation at my disposal.

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# CHAPTER I.

## The Determination of the Light-curves.

### § 1. *Method of Observation and Reduction.*

All the observations have been made after a somewhat modified Argelander method. The instruments used were the 10-inch refractor (aperture $a = 261$ mm ; focal length $f = 319$ cm ; magnifying-power $v = 94$ as a rule) and its 3-inch finder ($a = 74$ mm ; $f = 11.3$ cm ; $v = 22$). Some of the brightest stars have been observed with a binocular.

In the following pages the words „step" or „grade" (°) will be used instead of „Stufe".

Nijland, however, did not follow the original method of Argelander, but modified it into an *interpolation*-method [1]). If for instance the brightness of the variable $v$ was estimated as being between that of the comparison-stars $a$ and $b$, the observer first estimated the *ratio* of the differences $a—v$ and $v—b$ and then expressed these differences in steps. The observation may be recorded as $a\,1\,v\,2\,b$, or $a\,1\frac{1}{2}\,v\,3\,b$, or $a\,2\,v\,4\,b$ according to the estimation in steps. These records are not identical ; the interpolated magnitude of $v$, however, is in all cases the same, viz. $v = a + \frac{1}{3}(b—a)$.

The *ratio* of the differences remains the essential thing and for that reason the magnitude of the variable must be found by interpolation. Not infrequently, however, 1 or 3 comparison-stars have been used instead of 2 ; in these cases the magnitude of the variable was found by using the derived step-value (p. 3).

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1) A.N. **154**, 413 (1901).
See also : Recherches Astronomiques de l'Observatoire d'Utrecht III, **14** (1908).

If photometric magnitudes could be considered as absolutely correct, this interpolation would lead to PICKERING's fractional method. Unfortunately, however, photometric magnitudes are far from being correct, and they may, and should, to a certain degree, be controlled and, if necessary, corrected by the grade-estimates, so as to accord with the individual conception of the observer. After a great number of observations, the intervals between the comparison-stars, given in steps, may furnish sufficient data to revise the photometric magnitudes (see below).

This modified method is undoubtedly superior, not only to the old ARGELANDER method, which obliged the observers to stick to a constant step-value (although this value is certainly not constant), but also to the purely fractional method, which wholly ignores the step and assigns absolute accuracy to instrumental photometry.

As far as possible the photometric magnitudes of the comparison-stars were taken from the Harvard Photometry [1]) (HP), while magnitudes taken from other sources have been reduced to it. Originally NIJLAND used the following method of correcting the photometric magnitudes with the aid of the step-scale (i. e. the list of step-differences with the faintest comparison-star): The photometric magnitudes were plotted on squared-paper as abscissas and the step-scale as ordinates, 5 mm, say, representing, $0^{m}.1$ en $1°$ respectively ; and a straight line was drawn passing as nearly as possible through the points plotted. The magnitudes of the comparison-stars finally to be adopted were now obtained by dropping perpendiculars on this line from the plotted points.

It is clear that in this way equal weight is attributed to the photometrically determined magnitudes and to the grade-estimates ; obviously the estimated step-intervals are not used as they are given by the observational material, but have been modified so as to suit the photometric values. The slope of the straight line gives the photometric value of 1 step for the star in question.

As an example to illustrate this we choose *V* 23 = *SW Cygni* [2]).

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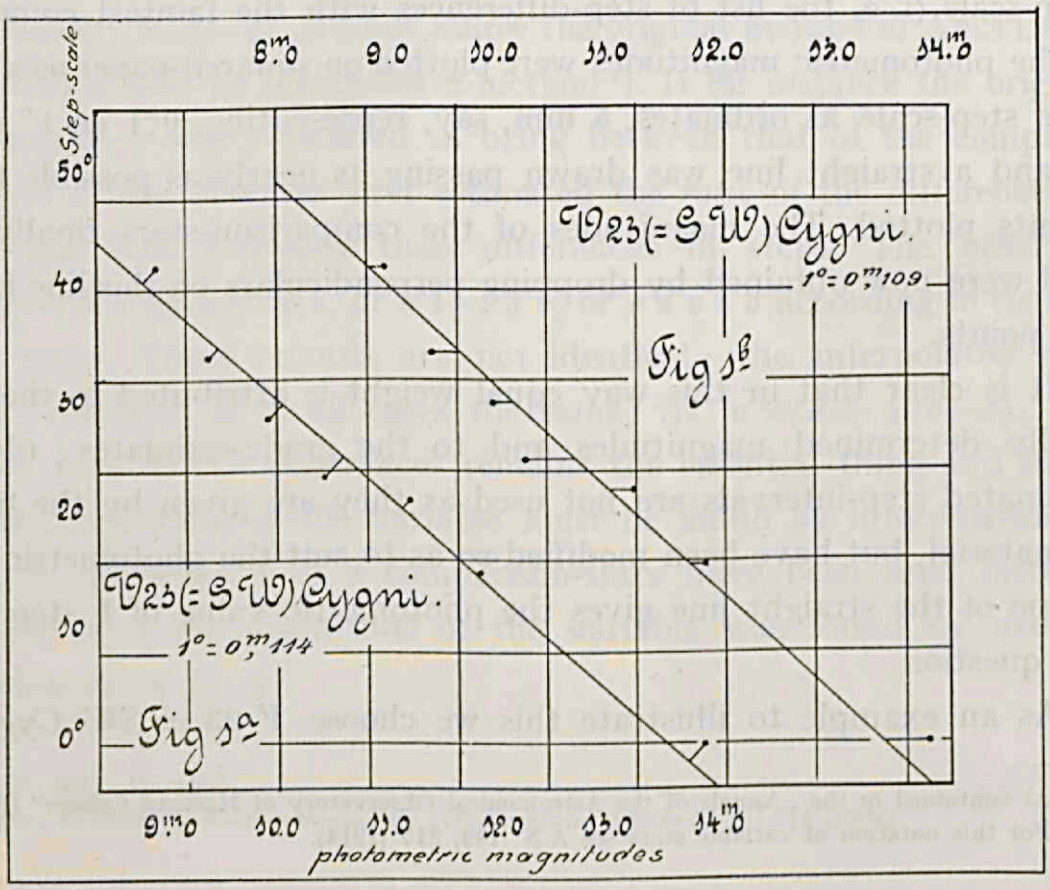
[1]) As contained in the ,,Annals of the Astronomical Observatory of Harvard College'' (H.A.).
[2]) For this notation of variable stars see A.N. **199**, 215 (1914).

TABLE 1.

| Comp. star | HP | Step-scale | After adjustment | |
|---|---|---|---|---|
| *a* | 8$^m$91 | 42°07 | 8$^m$80 | 40°8 |
| *b* | 9.39 | 33.86 | 9.48 | 34.9 |
| *c* | 9.92 | 28.70 | 10.04 | 30.0 |
| *d* | 10.42 | 23.48 | 10.58 | 25.3 |
| *e* | *11.18* | 21.28 | 11.12 | 20.6 |
| *f* | 11.80 | 14.80 | 11.79 | 14.8 |
| *k* | *13.79* | 0.00 | 13.64 | -1.8 |

The photometric magnitudes of the comparison-stars in the second column have been taken from H.A. **74**, except those of *e* and *k* whose magnitudes were derived from the limits of vision for the two instruments used[1]) (finder and telescope). The grade-estimates yield the scale given in the 3rd column. In Fig. 1*a* these two values have been plotted for each comparison-star ; through the points thus obtained a straight line has been drawn and perpendiculars have been dropped on it.




1) See on this subject : A.N. **205**, **233** (1917).

In the fourth column we find the magnitudes as they have been adopted for the comparison-stars after the adjustment described, whereas the 5th column shows how the step-scale is modified. It follows from the slope of the straight line, that 40 light-grades correspond to $4^{m}.56$, so that for this case $1^{\circ} = 0^{m}.114$.

As a matter of fact, however, the result obtained in this way is in many cases far from satisfactory, either on account of the small number of points used, or owing to the fact that the plotted points often strongly deviate from a straight line.

The magnitudes of the comparison-stars once having been adopted, the brightness of the variable is interpolated [1]). The corresponding phase is taken from the elements at hand. As a rule the series of observations was not closed until at least 250—400 estimates had been obtained. They were put in order of phase and combined into normals of 12 estimates each. These normals were then plotted and a smooth ,,*light-curve*'' [2]) was drawn through the points obtained, satisfying as well as possible the following conditions:

1°) the algebraic sum of the residuals $\delta$ should be zero: $\Sigma\,\delta = 0$;

2°) positive and negative residuals should be equal in number;

3°) the number of recurrences of sign should be equal to the number of changes of sign;

4°) the mean error $\varepsilon_0 = \sqrt{\frac{[\delta^2]}{m-\mu}}$ should correspond to the mean error $\varepsilon_1$ found in computing the normals. The number of parameters $\mu$ of the curve has been somewhat arbitrarily chosen, viz.:

$\mu = 6$ for curves without a stationary minimum;

$\mu = 8$ ,, ,, with ,, ,, ,,

Though NIJLAND himself has already derived light-curves for several eclipsing binaries observed by him, these results cannot, for reasons explained in the following paragraphs, be regarded as final. We shall therefore

1) See for the method of accounting for atmospheric extinction: Rech. Astr. de l'Obs. d'Utr. VIII, Erste Abt., No. 14.

2) In what follows the term ,,*light-curve*'' will be used for the curve giving the magnitude of the binary, and the term ,,*intensity-curve*'' for the curve giving the loss of light-intensity of the system.

here restrict ourselves to four systems (see chapter IV) whose light-curves have been revised according to the principles of § 4.

The results are:

1°) $\Sigma\,\delta = -0^m.05$;
2°) number of positive residuals = 35,
,, ,, negative ,, = 35,
,, ,, zeros = 14;
3°) number of recurrences of sign = 27,
,, ,, changes ,, ,, = 39;
(results of 4 systems combined.)
4°) $\varepsilon_0 = 0^m.031$,
$\varepsilon_1 = 0^m.033$.
(mean of 4 systems.)

It will interest the reader that for 17 Cepheids, treated in exactly the same way, Nijland found the following figures:

1°) $\Sigma\,\delta = +0^m.05$;
2°) number of positive residuals = 135,
,, ,, negative ,, = 129,
,, ,, zeros = 86;
3°) number of recurrences of sign = 111,
,, ,, changes ,, ,, = 153;
(results of 17 Cepheids combined.)
4°) $\varepsilon_0 = 0^m.031$,
$\varepsilon_1 = 0^m.029$.
(mean of 17 Cepheids).

We may remark that, the mean error $\varepsilon_1$ of a normal being about $0^m.030$, the mean error $\varepsilon$ of a single observation comes out as $0^m.030\sqrt{12} = 0^m.10$.

§ 2. *On the accuracy of the Photometric Magnitudes of the HP.*

We regret to state that this accuracy leaves rather much to be desired. As to H.A. **14** and H.A. **24** Müller and Kempf already pointed out [1]), that the accuracy of the photometric determinations suffered from the haste in which they were made. This is also proved by the large number of „discordant observations” mentioned by Pickering. Moreover Pickering was not quick in rejecting an estimate.

The magnitudes given in H.A. **14** are based on 3 observations, made

[1]) Publ. des Astrophys. Obs. zu Potsdam **9**, 122 and 491 (1894).

on 3 different nights. Of these the mean was simply taken when the measurements *inter se* did not differ more than one magnitude. Only when this was not the case, four more observations were made and the mean of the 7 values was adopted as a final result. A measurement was only rejected when it differed more than one magnitude from the mean of the other measurements.

The magnitudes mentioned in H.A. **24** are the mean of 2 observations, but here a limit of $0^m.6$ has been adopted as the allowable difference between the two observations [1]). If these differed $0^m.6$ or less the mean of the two values was accepted as the observed magnitude of the star. If the difference was greater, 3 more observations were made and the mean of those 5 observations was then accepted as the magnitude, unless one of the observations differed by $0^m.6$ or more from the mean of the other four ; in this case the last mean was accepted.

The photometric observations mentioned in H.A. **44** and in H.A. **45** are more accurate, but here too great differences between the individual measurements occur. In H.A. **44** [2]) at least 3 measurements have been taken of each star, and in H.A. **45** [3]) at least two. The residuals obtained by subtracting the mean magnitude from the results of the individual measures are still often considerable. As a rule residuals greater than $0^m.65$ have been rejected, except in the case of a few stars of great Southern declination.

H.A. **70** and H.A. **74** give photometric magnitudes of fainter stars and of special groups of stars. In H.A. **70** [4]) 3 measures have usually been taken of each star, 4 or sometimes 5 when the results of those 3 measures did not agree and, besides, in the case of standard-stars. Here again discrepancies of $0^m.6$ and more between the individual maesurements are not exceptional. In the case of the stars also occurring in H.A. **24** the difference has been given between the photometric magnitudes adopted in these two catalogues ; among these differences values of $0^m.5$ and more are by no means infrequent. H.A. **74** contains various groups of stars, down to the 13th magnitude ; the

[1]) H.A. **24**, 3.
[2]) H.A. **44**, 4.
[3]) H.A. **45**, 5.
[4]) H.A. **70**, 2.

method of observation is substantially the same as in H.A. **70** ; every star has been measured 1—6 times, but 3 times as a rule. In the case of stars also occurring in H.A. 50 and H.A. 54 (both being compilations of previous results), the differences with the values occurring in these volumes have been communicated. These differences too amount often to $0^m.5$ and more.

In H.A. **63** part II we find a list of magnitudes of comparison-stars for 279 variables. These magnitudes, however, have not been determined by accurate photometric methods. In the case of the brighter stars the photometric magnitudes have been taken from H.A. **50** and H.A. **54** ; in the case of the fainter stars from H.A. **74**. Moreover the intervals between the successive stars in the sequence were estimated in grades on 3 nights and the means taken. The successive sums were then found, assuming the light of the brightest star to be 0.0 grades. Points were next plotted with the photometric magnitudes as abscissas and the estimates in steps as ordinates. A smooth curve was drawn through these points and the magnitude read from the curve. The difference between the photometric magnitude and that derived from the means of the estimates has been given. This series of differences contains large values of an irregular character. For instance in the case of *V = XX Sagittarii* (p. 174) : $-0^m.34$ ; $+0^m.60$ ; $-0^m.27$ ; $+0^m.22$ ; $-0^m.29$ ; $+0^m.11$ ; $-0^m.22$.

Systematic differences between the two methods of observation are sure to occur, but still from the above considerations it appears that the accuracy of photometric determinations is far from unimpeachable.

In order to obtain a general idea of the accuracy of the Harvard catalogues we have computed the average mean errors $\varepsilon$ and $\varepsilon_0$ (of a single measurement and of a catalogue-value respectively), applying the formulas $\varepsilon^2 = \frac{[\delta^2]}{m-1}$ and $\varepsilon^2_0 = \frac{[\delta^2]}{m(m-1)}$ to 10 pages arbitrarily chosen in H.A. **14**, **24**, **44** and **45** (each containing about 450 stars) and to 20 pages (about 1000 stars) in H.A. **70**. Table 2 gives the mean results in the 6th row, together with the corresponding values for the Potsdam Photometric Durchmusterung (PD) [1]).

[1]) Publ. des Astrophys. Obs. zu Potsdam **9**, 491 (1894).

TABLE 2.

| Catalogue | $\varepsilon$ | $\varepsilon_0$ | Instrument |
|---|---|---|---|
| H.A. **14** | $0^m.22$ | $0^m.111$ | 2-inch meridian photometer |
| H.A. **24** | 0.19 | 0.133 | 4- ,, ,, ,, |
| H.A. **44** | 0.15 | 0.084 | 4- ,, ,, ,, |
| H.A. **45** | 0.14 | 0.085 | 4- ,, ,, ,, |
| H.A. **70** | 0.19 | 0.110 | 12- ,, ,, ,, |
| means | $0^m.18$ | $0^m.105$ | |
| PD | 0.084 | 0.059 | |

As we have seen the accuracy of the photometric magnitudes of the HP leaves something to be desired; it is obvious that, with an average m.e. of $0^m.105$ of a catalogue-value, errors of $0^m.2$ and even of $0^m.3$ cannot be exceptional. A simple calculation shows that e.g. in H.A. **24** with a total of 6700 we may expect no fewer than 800 catalogue-values to have an error $> 0^m.2$ ; 400 to have an error $> 0^m.25$ and 175 to have an error $> 0^m.3$. Of course, for stellar statistics the photometric material collected at Harvard is invaluable, but in special cases, as in our present investigation of the light-curves of eclipsing binaries the utmost care should be taken when basing a light-curve upon the Harvard magnitudes and we should certainly not follow the advice of the late Professor PICKERING to use the magnitudes given in H.A. **63**. On the contrary, since the magnitudes of the comparison-stars taken from the HP may sometimes differ to a large extent from the true values, the magnitudes of the variable, if based on them without further discussion, may be very inaccurate. The results to which this may lead, will be clear from the following two examples :

1°). When the observations of *V* 10 = *RR Delphini* were plotted the resulting light-curve was so far from smooth that it was wholly incompatible with any physical representation. As a matter of fact, both the descending and the ascending branches of the minimum showed a large hump. It was only after new photometric magnitudes for the comparison-stars had been determined at the Utrecht Observatory, one of them deviating considerably from the value first applied, that a new reduction of the observations led to a perfectly smooth light-curve.

2°) The first and most important part of RUSSELL's method for determining the orbital elements of the system is the derivation from the light-curve of the ratio $k$ of the radii of the two components. This value of $k$ must lie between 0 and 1. Every point of the light-curve may yield a value of $k$. Now the values of $k$ often proved to differ systematically from each other according to the points chosen ; so for instance in the case of $V$ 23 = *SW Cygni* and $V$ 25 = *SY Cygni* ; it also occurred that parts of the light-curve did not produce any value of $k$ at all. In connection with what follows in § 7, the possibility should be borne in mind, however, that these discrepancies may also be wholly or partly due to the method employed. When the magnitudes of the comparison-stars had been modified, however, so as to bring them into accordance with the step-scale (see § 4) it was surprising to see how well the theory of eclipses could be applied to the newly obtained light-curve in cases where such an application to the original curve had been impossible.

We subjoin the results for $V$ 23 = *SW Cygni* as an example :

TABLE 3.

| Comp. star | H.A. | H | H′ | H.A.—H′ |
|---|---|---|---|---|
| *a* | $8^m.91$ | $8^m.80$ | $8^m.80$ | $+0^m.11$ |
| *b* | 9.39 | 9.48 | 9.64 | —0.25 |
| *c* | 9.92 | 10.04 | 10.18 | —0.16 |
| *d* | 10.42 | 10.58 | 10.72 | —0.25 |
| *e* | *11.18* | 11.12 | 10.96 | +0.22 |
| *f* | 11.80 | 11.79 | 11.69 | +0.11 |
| *k* | *13.79* | 13.64 | 13.40 | +0.39 |

The first three columns of table 3 are identical with the 1st, 2nd and 4th of table 1 (p. 4) ; consequently column H gives the magnitudes of the comparison-stars after the adjustment described in § 1. Column H′ gives the magnitudes modified according to the step-scale (see § 4). The last column gives the differences between the magnitudes of the comparison-stars taken from H.A. **74** and those modified according to the step-scale.

The values of $k$ found from various points of the light-curve by Russell's method are given in table 4. This method, to be described in § 6, starts either from the supposition of a uniform distribution of intensity on both disks (U-hypothesis) or from the hypothesis of a complete darkening towards the limb (D-hypothesis). The first two rows of the table were derived from light-curve I, which was based on the magnitudes of the comparison-stars mentioned in the third column of table 3; the last two rows were derived from light-curve II, based on the magnitudes in the fourth column of that table. The notes of interrogation indicate that in these cases no values of $k$ are found between 0 and 1 (U-hyp.) or between 0.2 and 1 (D-hyp.).

TABLE 4.

| | | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.70 | 0.80 | 0.95 | 0.98 | 0.99 | 1.00 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Curve I | $k$ | 0.42 | 0.40 | 0.36 | 0.33 | 0.25 | 0.15 | ? | ? | ? | ? | ? | ? | U-hyp. |
| | $k$ | 0.40 | 0.50 | 0.50 | 0.50 | 0.45 | 0.37 | 0.22 | ? | ? | ? | ? | 0.34 | D-hyp. |
| Curve II | $k$ | 0.27 | 0.30 | 0.27 | 0.28 | 0.23 | 0.20 | 0.16 | 0.00 | 0.11 | 0.10 | 0.13 | 0.18 | U-hyp. |
| | $k$ | 0.24 | 0.39 | 0.40 | 0.44 | 0.42 | 0.41 | 0.40 | 0.22 | 0.32 | 0.40 | 0.46 | 0.55 | D-hyp. |

## § 3. *Revised Reduction of the Grade-estimates to the Photometric Scale.*

The difficulties which have been mentioned in the foregoing paragraph, may be due to the fact that in the derivation of magnitudes for the comparison-stars equal weight has been assigned to the photometric values of the HP and to the estimated light-steps. Therefore we decided to make a revised reduction to the photometric scale, laying full weight upon the observed steps, so as to bring the HP values into accordance with them. However, before leaving the apparently firm ground of photometry in order to trust ourselves to a scale of steps, we should subject the latter to a closer investigation. Then the following question at once presents itself: what is the value of the step, expressed in magnitudes and what factors influence this value? As such we may consider:

1°) the instruments used;
2°) the magnitudes of the stars observed;
3°) the number of steps estimated;
4°) the colour of the stars.

TABLE 5.

STEP-VALUE AND MEDIAN MAGNITUDE.

| | Variable | | Median Magnitude | Step-value |
|---|---|---|---|---|
| I | *Algol* | | 2$^m$9 | 0$^m$077 |
| | *λ Tauri* | | 3.8 | 0.090 |
| | *δ Cephei* | * | 3.9 | 0.069 |
| | *η Aquilae* | * | 4.0 | 0.087 |
| | *β Lyrae* | | 4.1 | 0.082 |
| | *ζ Geminorum* | * | 4.2 | 0.099 |
| II | *δ Librae* | | 5.5 | 0.092 |
| | *RT Aurigae* | * | 5.5 | 0.115 |
| | *S Sagittae* | * | 5.6 | 0.071 |
| | *T Vulpeculae* | * | 5.8 | 0.087 |
| | *Y Sagittarii* | * | 5.8 | 0.082 |
| | *R Canis Majoris* | | 6.0 | 0.103 |
| | *U Ophiuchi* | | 6.2 | 0.080 |
| | *SU Cassiopeiae* | * | 6.2 | 0.080 |
| | *T Monocerotis* | * | 6.2 | 0.096 |
| | *X Cygni* | * | 6.4 | 0.071 |
| III | *RZ Cassiopeiae* | | 7.1 | 0.092 |
| | *Z Herculis* | | 7.5 | 0.091 |
| | *RS Vulpeculae* | | 7.7 | 0.103 |
| | *TV Cassiopeiae* | | 7.8 | 0.119 |
| | *Z Vulpeculae* | | 7.8 | 0.101 |
| | *Y Cygni* | | 7.8 | 0.111 |
| | *U Sagittae* | | 8.0 | 0.092 |
| | *U Cephei* | | 8.0 | 0.098 |
| | *U Coronea Borealis* | | 8.1 | 0.087 |
| | *TW Draconis* | | 8.8 | 0.107 |
| | *SZ Cygni* | * | 9.2 | 0.091 |
| | *RY Persei* | | 9.5 | 0.093 |
| VI | *RT Lacertae* | | 10$^m$0 | 0$^m$111 |
| | *TT Lyrae* | | 10.2 | 0.094 |
| | *SW Draconis* | * | 10.4 | 0.078 |
| | *RW Ursae Majoris* | | 10.5 | 0.107 |
| | *RV Ursae Majoris* | * | 10.5 | 0.099 |
| | *U Scuti* | | 10.7 | 0.087 |
| | *RW Geminorum* | | 10.7 | 0.110 |
| | *SW Cygni* | | 10.8 | 0.114 |
| | *WZ Cygni* | | 10.8 | 0.125 |
| V | *RR Delphini* | | 11.0 | 0.115 |
| | *RT Persei* | | 11.1 | 0.089 |
| | *Z Persei* | | 11.1 | 0.106 |
| | *W Delphini* | | 11.1 | 0.106 |
| | *RR Draconis* | | 11.2 | 0.113 |
| | *ZZ Cygni* | | 11.3 | 0.093 |
| | *SX Persei* | * | 11.3 | 0.082 |
| | *XX Cygni* | * | 11.4 | 0.080 |
| | *SY Andromedae* | | 11.4 | 0.118 |
| | *RZ Lyrae* | * | 11.6 | 0.108 |
| | *WW Cygni* | | 11.6 | 0.099 |
| | *RV Persei* | | 11.6 | 0.123 |
| | *UW Cygni* | | 11.6 | 0.126 |
| | *Z Draconis* | | 11.6 | 0.124 |
| VI | *SY Cygni* | | 12.2 | 0.133 |
| | *TT Andromedae* | | 12.3 | 0.109 |
| | *RZ Camelopardalis* | * | 12.6 | 0.118 |
| | *VV Cygni* | | 13.4 | 0.120 |

[1]) Unless the reverse has been mentioned, the variables of the *β Lyrae-type* are tacitly included. When the stars of this latter type are separately treated, we shall indicate them for shortness' sake by the name of „*Lyrids*".

From the very beginning it was plain that the step-value depends on the stellar magnitude : it increases for fainter stars. As stated on p. 3 a discussion of the comparison-stars of a variable may yield the value of 1 step for any particular case. For this discussion eclipsing binaries[1]) for which the series of observations had been closed were available, and likewise 17 Cepheids. The results have been collected in table 5.

The first column gives the name of the variable ; the stars marked with an asterisk are Cepheids. The second column contains the median-magnitude (mean of the maximum and minimum brightness). The third column gives the step-value derived for each variable. These data have been joined into six groups. Table 6 gives the mean median-magnitude and the mean step-value for each group.

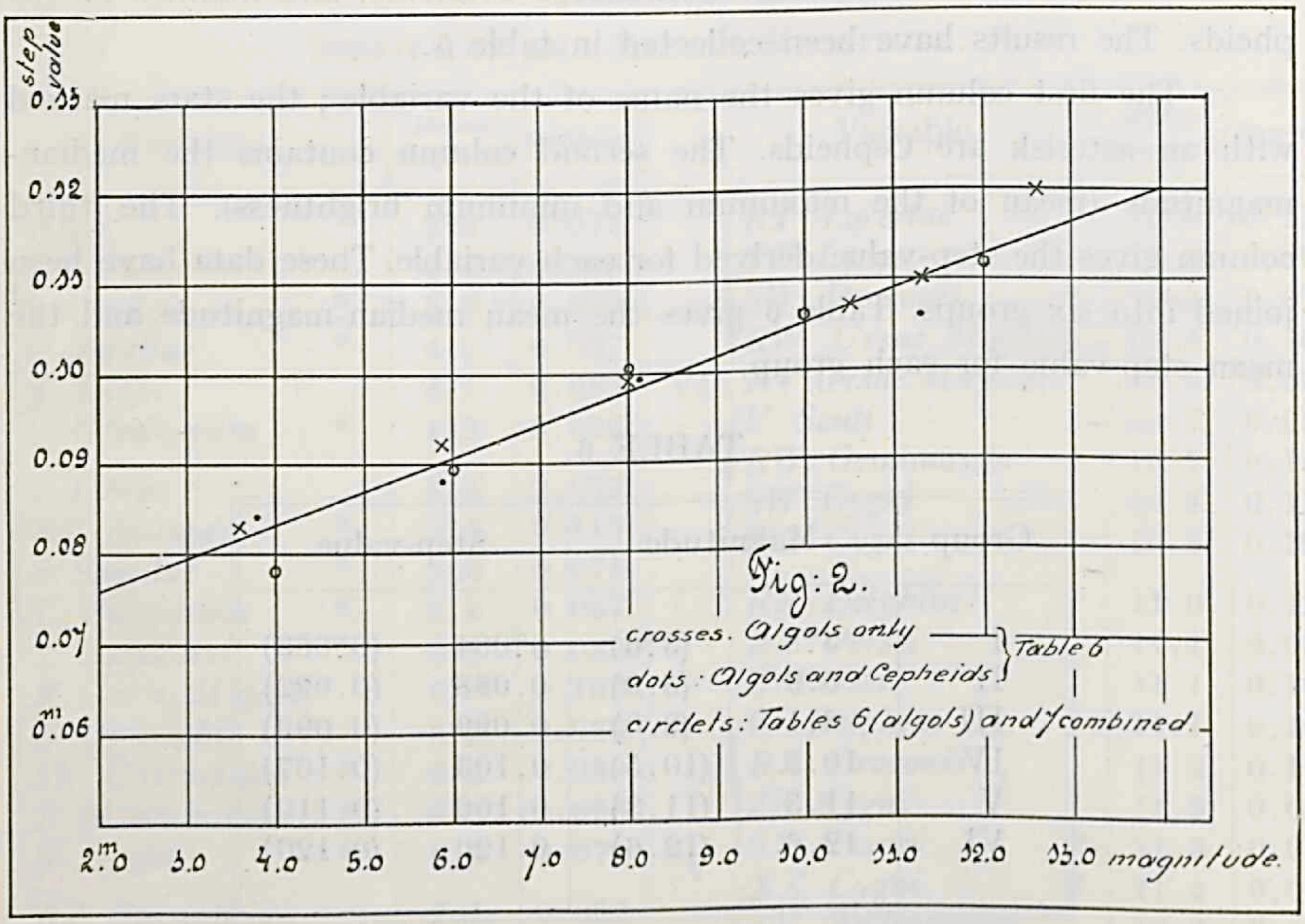
TABLE 6.

| Group | Magnitude | | Step-value | |
|---|---|---|---|---|
| I | $3^m\!.8$ | ($3^m\!.6$) | $0^m\!.084$ | ($0^m\!.083$) |
| II | 5.9 | (5.9) | 0.088 | (0.092) |
| III | 8.1 | (8.0) | 0.099 | (0.099) |
| IV | 10.5 | (10.5) | 0.103 | (0.107) |
| V | 11.3 | (11.3) | 0.106 | (0.110) |
| VI | 12.6 | (12.6) | 0.120 | (0.120) |

The values between brackets are obtained by excluding the Cepheids. For both cases it is evident that there is a distinct increase of the step-value with decreasing brightness. The last figure (group VI) is rather uncertain on account of the scarcity of material in this group and of the difficulty of estimating steps in an interval near the limit of vision.

In Fig. 2 the dots indicate the points we obtain if both Algols and Cepheids are included, and the crosses the points we get if the Cepheids are excluded. A glance at this Fig. 2 will show that the correlation may be considered to be linear.

NIJLAND obtained the same result in a different way. Hitherto it was generally believed that the step-value increased with the number of steps estimated[1]). NIJLAND himself found strong evidence for this variability in a discussion of his observations of *Algol* [2]). In 1920, however, while studying many thousands of his estimates, to his great astonishment he came to wholly different results, which may be summarized as follows:




1°) The step-value does not depend upon the estimated number of steps[3]); from this it follows that when a variable has been compared with a comparison-star *via* another comparison-star as an intermediate, such an estimate may be reduced with the same step-value as the directly estimated intervals.

2°) If an interval has been estimated in two different instruments, there is no perceptible influence of the instrument used; i. e. the step-value is the same for telescope, finder and binocular; therefore it is unnecessary

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1) Hagen: Die veränderlichen Sterne, II, p. 200.

2) A.N. **154**, 413 (1901).

3) It is to be emphasized that this number never exceeded 8.

to treat the 3 instruments separately or to reduce them to each other [1]).

3°) For one and the same instrument the step-value depends on the brightness of the stars included [2]). For the study of this correlation two sources of material were available, viz.:

*a*) a comparatively small number of directly estimated step-intervals of comparison-stars;

*b*) a large amount of indirect estimations, the variable itself being the link between the comparison-stars. On account of what has been found sub 1°) an observation of this kind, say $a\,m\,v\,n\,b$, yields the step-interval between $a$ and $b$, viz. $m + n$, as if $a$ and $b$ had been directly compared.

Both cases were treated separately and the second was given threefold weight, in general, on account of the large number of facts included. The results are given in table 7.

TABLE 7.

| | 2ᵐ0—3ᵐ0 | 3ᵐ0—5ᵐ0 | 5ᵐ0—7ᵐ0 | 7ᵐ0—9ᵐ0 | 9ᵐ0—11ᵐ0 | 11ᵐ0—12ᵐ0 | 12ᵐ0—13ᵐ0 |
|---|---|---|---|---|---|---|---|
| 1°= | 0ᵐ55 | 0ᵐ74 | 0ᵐ087 | 0ᵐ100 | 0ᵐ107 | 0ᵐ115 | 0ᵐ100 |

The last figure is very doubtful on account of the scarcity of the material used. We have combined the results of table 7 as well as possible with those of table 6 (Cepheids excluded) into 5 groups, represented in Fig. 2 by circlets. It is seen that the connection between step-value and magnitude may be represented by a straight line.

Finally it seems worth while to consider whether the step-value depends on the amplitude of the light-variation. To answer this question, we have with the aid of fig. 2, reduced the step-value following from the discussion of the comparison-stars (p. 3) to the value, which would have been

[1]) The equivalence of two instruments does, however, not hold for such stars as are near the limit of vision of the smaller instrument.

[2]) This statement, which might lead to a contradiction with that mentioned sub. 2°, will be submitted to a closer investigation later on.

obtained if the median-magnitude had been $8^{\mathrm{m}}0$. Thus the step-values have all been reduced to the same magnitude (table 8).

TABLE 8.

| Star | Median Magnitude | Reduced Step-value | Range | Star | Median Magnitude | Reduced Step-value | Range |
|---|---|---|---|---|---|---|---|
| *Algol* | $2^{\mathrm{m}}9$ | $0^{\mathrm{m}}096$ | $1^{\mathrm{m}}30$ | *RT Lacertae* | $10^{\mathrm{m}}0$ | $0^{\mathrm{m}}103$ | $1^{\mathrm{m}}00$ |
| *λ Tauri* | 3.8 | 0.106 | 0.40 | *TT Lyrae* | 10.2 | 0.087 | 2.10 |
| *δ Cephei* * | 3.9 | 0.081 | 0.60 | *SW Draconis* * | 10.4 | 0.071 | 0.80 |
| *η Aquilae* * | 4.0 | 0.102 | 0.70 | *RW Ursae Majoris* | 10.5 | 0.097 | 1.10 |
| *β Lyrae* | 4.1 | 0.096 | 0.80 | *RV Ursae Majoris* * | 10.5 | 0.090 | 1.00 |
| *ζ Geminorum* | 4.2 | 0.116 | 0.40 | *U Scuti* | 10.7 | 0.079 | 1.10 |
| *δ Librae* | 5.5 | 0.101 | 1.00 | *RW Geminorum* | 10.7 | 0.100 | 1.90 |
| *RT Aurigae* * | 5.5 | 0.127 | 1.10 | *SW Cygni* | 10.8 | 0.103 | 2.90 |
| *S Sagittae* * | 5.6 | 0.078 | 1.00 | *WZ Cygni* | 10.8 | 0.113 | 1.00 |
| *T Vulpeculae* * | 5.8 | 0.095 | 0.80 | *RR Delphini* | 11.0 | 0.103 | 1.40 |
| *Y Sagittarii* * | 5.8 | 0.089 | 0.80 | *RT Persei* | 11.1 | 0.080 | 1.30 |
| *R Canis Majoris* | 6.0 | 0.111 | 0.70 | *Z Persei* | 11.1 | 0.095 | 2.30 |
| *U Ophiuchi* | 6.2 | 0.086 | 0.60 | *W Delphini* | 11.1 | 0.095 | 2.20 |
| *U Cassiopeiae* * | 6.2 | 0.086 | 0.40 | *RR Draconis* | 11.2 | 0.101 | 3.80 |
| *T Monocerotis* * | 6.2 | 0.104 | 1.00 | *ZZ Cygni* | 11.3 | 0.082 | 1.10 |
| *X Cygni* * | 6.4 | 0.076 | 0.70 | *SX Persei* * | 11.3 | 0.073 | 0.70 |
| *RZ Cassiopeiae* | 7.1 | 0.095 | 1.50 | *XX Cygni* * | 11.4 | 0.071 | 0.70 |
| *Z Herculis* | 7.5 | 0.093 | 0.60 | *SY Andromedae* | 11.4 | 0.104 | 1.60 |
| *RS Vulpeculae* | 7.7 | 0.104 | 0.80 | *RZ Lyrae* * | 11.6 | 0.095 | 1.30 |
| *TV Cassiopeiae* | 7.8 | 0.120 | 1.00 | *WW Cygni* | 11.6 | 0.087 | 3.40 |
| *Z Vulpeculae* | 7.8 | 0.102 | 1.40 | *RV Persei* | 11.6 | 0.108 | 2.60 |
| *Y Cygni* | 7.8 | 0.112 | 0.60 | *UW Cygni* | 11.6 | 0.111 | 2.70 |
| *U Sagittae* | 8.0 | 0.092 | 2.80 | *Z Draconis* | 11.6 | 0.109 | 2.40 |
| *U Cephei* | 8.0 | 0.098 | 2.40 | *SY Cygni* | 12.2 | 0.114 | 2.40 |
| *U Coronae Borealis* | 8.1 | 0.087 | 1.00 | *TT Andromedae* | 12.3 | 0.094 | 1.50 |
| *TW Draconis* | 8.8 | 0.104 | 2.10 | *RZ Camelopardalis* * | 12.6 | 0.100 | 1.00 |
| *SZ Cygni* * | 9.2 | 0.086 | 0.70 | *VV Cygni* | 13.4 | 0.099 | 0.80 |
| *RY Persei* | 9.5 | 0.088 | 2.40 | | | | |

The data of table 8 were grouped according to the amplitude, and for each group the mean step-value was deduced. We thus get table 9, in which the figures between brackets refer to the case of eclipsing binaries only.

This table does not show any influence of the amplitude on the step-value. Indirectly it appears once more, that the step-value does not depend upon the estimated number of steps.

TABLE 9.

| Range | Number | | Mean step-value | |
|---|---|---|---|---|
| $0^m0$—$0^m9$ | 19 | ( 8) | $0^m.092$ | ($0^m.101$) |
| 1.0—1.9 | 22 | (16) | 0.098 | (0.097) |
| 2.0 and greater | 14 | (14) | 0.099 | (0.099) |

Summarizing we find the following results from the preceding investigation :

1°) the step-value does not depend upon the estimated number of steps ;
2°) the step-value increases with decreasing brightness.

§ 4. *A new Determination of the Magnitudes of the Comparison-stars.*

In order to connect the magnitudes of the comparison-stars as closely as possible with the step-scale we proceeded as follows :

First the observed step-intervals in the sequence of comparison-stars were reduced with the aid of Fig. 2 to the median magnitude of the variable. By this reduction the change in the step-value during the process of the light-variation is taken into account ; obviously an appreciable effect is only to be expected in light-curves of very great range, e. g. *V* 45 = *WW Cygni*. Next this homogeneous step-scale and the photometric magnitudes taken from the HP were plotted in the usual way and through the points obtained a straight line was drawn. In a few cases, where the slope of the line could not be determined with certainty from the points available, these being too few in number, the „theoretical" step-value derived from Fig. 2 was taken into consideration. The difference with the process mentioned on p. 3 lies in the fact that we did not, this time, draw perpendiculars on the line, but lines parallel to the axis of abscissas, so that we leave the step-scale unaltered. In other words we consider it to be free from errors. Are we justified in doing so? It is true that in estimating steps, contrary to photometric determinations, the remembrance of previous estimates in the same interval might influence the results, but apart from this the step-method in the hands of an experienced observer need not be second to photometric determinations, which are no more free from systematic errors even of a sometimes inconceivable and complicate character [1]). In the HP according to p. 9 the mean error $\varepsilon$ of a single measurment is about $0^m.18$ and the mean error $\varepsilon_0$ of a catalogue-value about $0^m.10$ [2]). The m. e. ($\varepsilon$) of a grade-estimate is about $0^m.10$ (see p. 6). In general, therefore, a grade-estimate is not less accurate than a photometric determination. To this we may add :

1°) that the grade-estimates are usually very numerous (often 30—80 in the interval between 2 comparison-stars) ;

2°) the possibility of strong systematic personal differences in the appreciation of brightness between the observer and the author of the photometric catalogue, mostly due to the colour of the stars.

Especially this latter consideration justifies keeping the step-scale immodified. Except for the general course of the straight line — its slope, giving the step-value for each particular case — the photometric magnitudes have no longer been taken into account. Only if the number of grade-estimates in an interval is very small, and in some more special cases, we applied little corrections to the step-value, so as to lead to a closer correspondence with the individual photometric magnitudes.

As an example we once more take *V* 23 = *SW Cygni*.

The first four columns of table 10 are identical with those of table 1 (p. 4). Column 5 contains the observed step-intervals of the successive comparison-stars ; column 6 these intervals reduced to $11^m.0$ ; column 7 the new step-scale. Column 8 gives the magnitudes of the comparison-stars,

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1) See e.g. : Contr. from the Princeton Un. Obs. I : The Algol-system *RT Persei*, by R. S. Dugan.

2) According to a statement on the same page these values are much less for the PD, viz. $0^m.084$ and $0^m.059$.

Dugan gives in the Contr. from the Princeton Un. Obs. **5**, 29 (1920) for the probable error of a single observation of full weight in the case of *U Cephei*, *RT Persei*, *Z Draconis*, *RV Ophiuchi* and *RZ Cassiopeiae* an average of about $0^m.04$.

TABLE 10.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| *a* | $8^m91$ | 42°07 | $8^m80$ | 8°21 | 7°64 | 42°13 | $8^m80$ |
| *b* | 9.39 | 33.86 | 9.48 | 5.16 | 4.90 | 34.39 | 9.64 |
| *c* | 9.92 | 28.70 | 10.04 | 5.22 | 5.05 | 29.59 | 10.18 |
| *d* | 10.42 | 23.48 | 10.58 | 2.20 | 2.18 | 24.54 | 10.72 |
| *e* | *11.18* | 21.28 | 11.12 | 6.48 | 6.60 | 22.36 | 10.96 |
| *f* | 11.80 | 14.80 | 11.79 | 14.80 | 15.76 | 15.76 | 11.68 |
| *k* | *13.79* | 0.00 | 13.64 | | | 0.00 | 13.40 |

obtained in the above way; see also Fig. 1*b*, (p. 4) (the magnitude-scale is given at the top). In this case a step-value of $0^m109$ is found for the median magnitude ($11^m0$). The magnitudes of column 8 are those mentioned in the fourth column of table 3 (p. 10), with which a new light-curve has been derived. As already stated (p. 10) this curve gives much better values of *k* (see table 4).

# CHAPTER II.

## The Theory of Russell.

§ 5. *On the physical cause of the light-variation.*

Two theories have been given which might explain the light-variation of this group of variable stars:

1 *The spot-hypothesis.*

This explanation has first been treated by Zöllner [1]) and afterwards fully by Bruns [2]) and by Harting [3]). Bruns found that it is always possible, in an infinite number of ways, to assume spots on the body of a star located in such a way, that by its axial rotation the light-variation agrees within arbitrarily chosen limits with the observations.

2°) *The eclipse-theory.*

This explanation was originally given by Goodricke [4]). About a century later E. C. Pickering [5]) based upon this explanation a theory which, with various restrictions, enabled him to deduce the elements of the system β *Persei* from the light-curve. His example was followed by Harting [3]), Wilson [6]) and Blažko [7]), who expanded the theory and were able to drop some of the restrictions. Blažko even went a little further by introducing into his considerations the possibility of a diminution of light from the center towards the limb of the star's disk, an idea which had been previously [8]) suggested already.

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1) Zöllner: Photometrische Untersuchungen IV, §§ 71—80. Leipzig 1865.

2) W. Bruns: Bemerkungen über den Lichtwechsel der Sterne vom Algoltypus. Akad. Berlin, 1881. S.48.

3) Joh. Harting: Inaugural-Dissertation; München 1889.

4) John Goodricke: A Series of Observations on, and a Discovery of the Period of Algol. Phil. Trans. 1783.

5) E. C. Pickering: Dimensions of the fixed Stars, etc. Proceedings of the American Acad. of Arts and Sciences; Vol. XVI, 1881.

6) H. C. Wilson: Variable Stars of the Algol-type. Pop. Astr. 8; 113 (1900).

7) S. Blažko: Annales de l'Observatoire Astronomique de Moscou. 5; 76 (1911).

8) Rödiger: Untersuchungen über das Doppelsternsystem Algol; Königsberg, 1902.

After the spectrographic observations of VOGEL in the year 1889 [1]), it became at once evident that the true explanation of the phenomenon should be sought in the direction of the eclipse-theory. Finally in the year 1912 H. N. RUSSELL published an analytical method by which the orbital elements may be deduced from the light-curve. A brief summary of this theory is given in the next paragraph.

§ 6. *Summary of* RUSSELL'S *method.*

A. *Notations.*

RUSSELL has considered two extreme cases, viz.:

I. The star-disks are supposed to be „uniformly" illuminated. This case will be called the **„U-hypothesis"**.

II. The light of both disks is gradually decreasing from center to limb (where it has an intensity zero). This case will be called the **„D-hypothesis"**.

The cases actually observed will lie between these two extremes.

In explaining the theory we shall assume that the smaller star moves in a circular orbit around the larger. The radius of the orbit is taken as the unit of length and the total light of the system as the unit of light.

RUSSELL uses the following notations:

$P$ period of revolution in days;

$i$ inclination of the orbit, i.e. angle between line of sight and normal to the orbital plane. As a rule this angle lies between 75° and 90°;

$\tau$ time in days, measured from primary minimum;

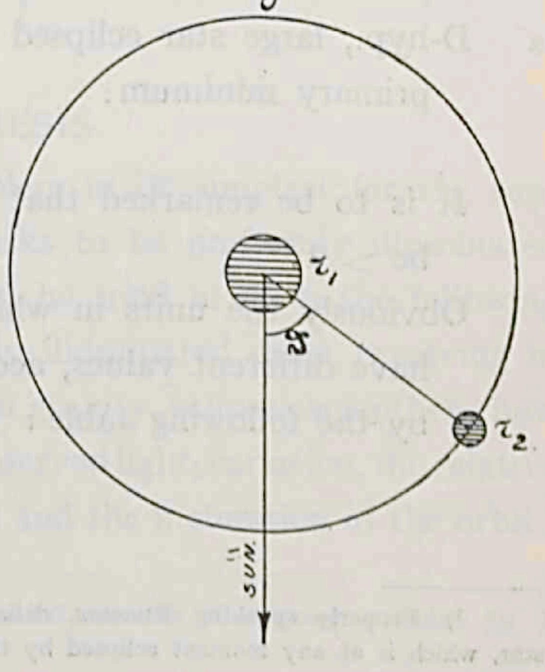
$\vartheta$ angle $\frac{2\pi}{p}t$, mean anomaly of smaller star in its orbit (see Fig. 3);

$\delta$ apparent distance of centers; } Unit:

$r_1$ radius of larger star; } radius

$r_2$ radius of smaller star; } of orbit.

Fig. 3.




1) A.N. 123, 289 (1890).

$k$ ratio of radii $= \frac{r_2}{r_1}$;

$p$ ratio, in the case of circular star-disks, of $\delta - r_1$ (distance from the center of the small disk to the circumference of the larger one) to $kr_1$ (radius of the small disk);

$\rho_1, \rho_2$ densities of large and small star respectively;

$I_1$ surface-brightness of larger star;

$I_2$ surface-brightness of smaller star;

$\gamma$ the ratio $\frac{I_2}{I_1}$; in the case of the D-hyp. $\gamma$ represents an average;

$L_1$ light of larger star;

$L_2$ light of smaller star;

$l$ light at any moment; hence $1 - l$ the loss of light at that moment;

$\lambda$ light at minimum; hence $1 - \lambda$ the loss of light at that minimum;

(for $L_1$, $L_2$, $l$, $\lambda$:) Unit: total light of the system: $L_1 + L_2 = 1$

**N.B.** The subscript 1 to the latter two quantities refers to the eclipse of the *small* star by the *large* one; the subscript 2 to the other eclipse.

$\alpha$[1]) U-hyp.:

$\alpha'$ D-hyp., small star eclipsed at primary minimum:

$\alpha''$ D-hyp., large star eclipsed at primary minimum:

(for $\alpha$, $\alpha'$, $\alpha''$:) loss of light at any moment expressed in terms of the loss at the moment of internal tangency (total or annular eclipse); this does not imply that internal tangency actually occurs in the eclipse under consideration.

It is to be remarked that $\alpha$ and $\alpha'$ must be $\leq 1$, whereas $\alpha''$ may also be $> 1$.

Obviously the units in which the quantities $\alpha$, $\alpha'$ and $\alpha''$ are expressed have different values, according to the cases considered, as is shown by the following table:

[1]) Properly speaking RUSSELL defines $\alpha$ as that part of the surface of the disk of the smaller star, which is at any moment eclipsed by the larger. Obviously the two definitions are equivalent.

TABLE 11.

| Hypothesis | Eclipse | Loss of light | Unit |
|---|---|---|---|
| U | Smaller star eclipsed | $\alpha$ | $L_2$ |
| U | Larger " " | $\alpha$ | $k^2 L_1$ |
| D | Smaller " " | $\alpha'$ | $L_2$ |
| D | Larger " " | $\alpha''$ | $L_1\, Q\, (k, 1)$ [1] |

$\alpha_0, \alpha_0'$ and $\alpha_0''$ the values of $\alpha$, $\alpha'$ and $\alpha''$ at the middle of the eclipse.

Since we have supposed the orbits to be circular, the values of $\alpha_0$ are in the U-hyp. the same for the two minima. In the D-hyp., however, this is not the case; here $\alpha_0'$ and $\alpha_0''$ are connected by the relation (12) of p. 35.

$\alpha_0 = 1$ if the eclipse is total or annular and $< 1$ if it is partial;
$\alpha_0' = 1$ if the eclipse is total and $< 1$ if it is partial;
$\alpha_0'' > 1$ if the eclipse is annular and $< 1$ if it is partial.

$n$ fraction of greatest loss of light in partial eclipses;
$x$ coefficient of darkening;
$a_1$ and $a_2$ semi-major axes;
$b_1$ and $b_2$ semi-minor axes;
$\varepsilon$ excentricity of meridian-section;
$z$ the quantity $\varepsilon^2 \sin^2 i$;
(the last four: In the case of ellipsoidal stars.)

## B. U-HYPOTHESIS.

Russell has first treated the problem in its simplest form [2]), supposing the orbit to be circular and the disks to be uniformly illuminated according to the cosine-law. In other words, he tried to solve the following problem: Two spherical stars with uniformly illuminated disks, revolving in circular orbits around their common center of gravity, eclipse each other; how to find, from considerations based upon the observed light-variation, the relative dimensions and brightness of the two stars and the inclination of the orbit?

[1]) See for this Q-function p. 35.
[2]) H. N. Russell: On the determination of the orbital elements of eclipsing variables. Ap. J. 35, 315 (1912).

In this problem we may distinguish four cases :

1°) Both primary and secondary light-variation have been observed and show a constant minimum-light.

2°) The primary light-variation shows a constant minimum-light ; the secondary minimum cannot be exactly determined or is imperceptible ; at least the observations have not revealed it.

3°) Both primary and secondary light-variation have been observed, but show no constant minimum light.

4°) Only the primary light-variation, showing no constant minimum light, has been observed.

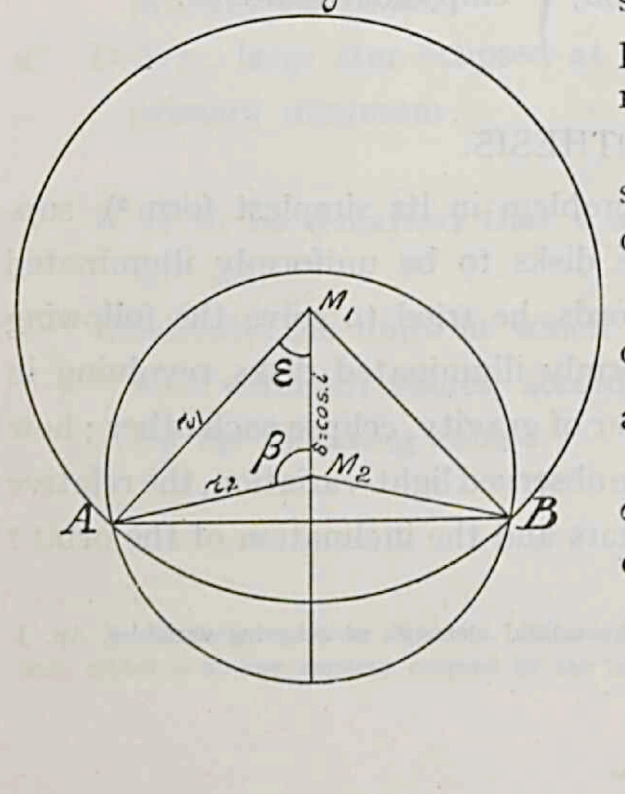
In the first and the second case we have to deal with total or annular eclipses ; as a rule we cannot decide between these two before a value of $k$ has been found.

First of all we shall have to pass from the light-curve to the intensity-curve (p. 5, note 2). If at any moment $l$ represents the intensity, $m$ the magnitude, $m_0$ the magnitude at the minimum, $C$ and $c$ two constants, we have :

$$L = C \times 2.512^{c-m}$$
$$1 = C \times 2.512^{c-m_0}$$
$$\overline{L = 2.512^{m_0-m}}$$

whence : log. $l = 0.4\ (m_0 - m)$.

Fig4.




This formule enables us to construct a table giving the loss of light corresponding to an increase $\Delta m$ in stellar magnitude (Table A at the end of this paper).

The part of the disk of the smaller star, obscured at any moment, is the sum of two circular segments (Fig. 4).

In $\Delta\ A M_1 M_2$ :

$$\cos \beta = \frac{r_2^2 + \delta^2 - r_1^2}{2 r_2 \delta} \text{ and } \cos \varepsilon = \frac{r_1^2 + \delta^2 - r_2^2}{2 r_1 \delta}$$

and further we readily find :

$$\alpha = \frac{\frac{\beta}{180} \pi r_2^2 - \frac{1}{2} r_2^2 \sin 2\beta + \frac{\varepsilon}{180} \pi r_1^2 - \frac{1}{2} r_1^2 \sin 2\varepsilon}{\pi r_2^2}$$

or :

$$a = \frac{\frac{\beta}{180} \pi k^2 + \frac{\varepsilon}{180} \pi - \frac{1}{2} k^2 \sin 2\beta - \frac{1}{2} \sin 2\varepsilon}{\pi k^2}$$

These formulas are not given by RUSSELL; we insert them because we shall have to refer to them later on. It at once appears, that the ratio $\alpha$ only depends on the ratios of the quantities $r_1$, $r_2$ and $\delta$, for instance on $\frac{r_2}{r_1} = k$ and $\frac{\delta}{r_1} = 1 + kp$. Therefore we may write $\alpha = f\,(k, \frac{\delta}{r_1})$.

For any given value of $k$ we may invert this function and write

$$\frac{\delta}{r_1} = \varphi\,(k, \alpha) \quad \text{or:} \quad 1 + kp = \varphi\,(k, \alpha).$$

Thus $p$ becomes a function of $k$ and $\alpha$, which has been computed and put in a tabular form by RUSSELL[1]) (Table I at the end of this paper). For every set of values of $k$ and $\alpha$ this table gives a value of $p$ and therefore of the function $\varphi$.

Further we derive from simple geometrical considerations:

$$d^2 = \cos^2 i + \sin^2 i \ \sin^2 \vartheta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

whence

$$\cos^2 i + \sin^2 i \ \sin^2 \vartheta = r_1^2 \ \{\varphi\,(k, \alpha)^2\} \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

An equation of this form may be deduced for any value of $\alpha$—consequently for any point of the light-curve. From several equations of the form (1$b$) we now must derive the three unknown quantities $k$, $r_1$ and $i$. This appears to be rather complicated; RUSSELL gives the following method:

Let $\alpha_1$, $\alpha_2$, $\alpha_3$ be any definite values of $\alpha$ and $\vartheta_1$, $\vartheta_2$, $\vartheta_3$ the corresponding values of $\vartheta$, which may be found from the light-curve. From the three corresponding equations of form (1$b$) we derive:

$$\frac{\sin^2 \vartheta_1 - \sin^2 \vartheta_2}{\sin^2 \vartheta_2 - \sin^2 \vartheta_3} = \frac{\{\varphi\,(k, \alpha_1)\}^2 - \{\varphi\,(k, \alpha_2)\}^2}{\{\varphi\,(k, \alpha_2)\}^2 - \{\varphi\,(k, \alpha_3)\}^2} = \psi\,(k, \alpha_1, \alpha_2, \alpha_3) \ldots\ldots\ldots\ldots (2)$$

The first member of this equation contains only known quantities. Now choose once for all two fixed values for $\alpha_2$ and $\alpha_3$, viz. $\alpha_2 = 0.6$ and $\alpha_3 = 0.9$, corresponding to two fixed points $a$ and $b$ on the light-curve. Thus $\psi$ becomes a function of $k$ and $\alpha_1$ only, and may be tabulated for suitable intervals in these two arguments. If, for shortness' sake, we put

$$A = \sin^2 \vartheta_2 \ \text{and} \ B = \sin^2 \vartheta_2 - \sin^2 \vartheta_3$$

equation (2) may be written:

$$\sin^2 \vartheta_1 = A + B\,\psi\,(k, \alpha_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

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1) Ap.J. 35, 333 (1912).

The value of $k$ is now determined as follows: for several — say 12 — values of $\alpha_1$, corresponding to as many points on the light-curve, we may read off from that curve the corresponding values of $\vartheta_1$, which by means of equation (3) furnish as many values of $\psi$. Then the $\psi$-table gives a value of $k$ for each set of values $\psi$, $\alpha_1$. By taking a suitably weighed mean of these values of $k$[1]), *a* theoretical light-curve can be obtained which passes through the fixed points $a$ and $b$, but will deviate more or less from the other points of the observed curve. By slight changes in the assumed positions of $a$ and $b$ (i. e. in the corresponding values of $\vartheta$, or of $\tau$ and therefore in those of $A$ and $B$), it is possible to obtain a computed curve which fits the whole course of the observed curve as well as possible. The criterion of this is that the upper (above $a$), the middle (between $a$ and $b$) and the lower (below $b$) parts of the observed curve will sensibly yield the same mean value of $k$[2]). The values of $k$ found with the different values of $\alpha_1$, may of course differ among each other to a certain extent.

This point and the possibilities which may arise from it will be more fully treated in § 7.

When once $k$ has been determined, we may with the aid of equation (3) find values $\vartheta'$ and $\vartheta''$ for the moment of the beginning of eclipse ($\alpha_1 = 0$) and that of the beginning of totality ($\alpha_1 = 1$). These computed values are more accurate than those estimated from the observed curve[3]). Since at these phases of the eclipse $\delta' = r_1 + r_2 = r_1 (1 + k)$ and $\delta'' = r_1 - r_2 = r_1 (1 - k)$ respectively, we may derive from (1*a*):

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[1]) The individual determinations of $k$ are of very different weight. Between $a$ and $b$ (that is for values of $\alpha_1$ between 0.6 and 0.9) $\psi$ changes very slowly with $k$. At the beginning and end of the eclipse the stellar magnitude changes very slowly with the time, and hence, by (3), with $\psi$. The corresponding parts of the curve are therefore ill adapted to determine $k$. For the first approximation it is well to give the values of $k$ derived from values of $\alpha_1$ between 0.95 and 0.99 and between 0.4 and 0.2 double weight (provided the corresponding parts of the curve are well fixed by observation). The time of beginning or end of eclipse cannot be read with even approximate accuracy from the observed curve and should not be used at all in finding $k$. The beginning or end of totality may sometimes be determined with fair precision, but does not deserve as much weight as the neighbouring points on the steep part of the curve. (Ap. J. **35**, p. 322).

[2]) If further refinement is desired, it can most easily been obtained by plotting the light-curve for two values of $k$ and comparing with a plot of the observations. This will rarely be necessary.

[3]) The latter values played an important part in the older treatment of the problem by Wilson and others. Russell too has used them — though he gave them less weight — in the determination of $k$ from various points of the light-curve.

$$r_1^2\ (1+k)^2 = \cos^2 i + \sin^2 i \sin^2 \vartheta'$$

$$r_1^2\ (1-k^2) = \cos^2 i + \sin^2 i \sin^2 \vartheta''$$

These equations finally give the elements $r_1$ and $i$.

Moreover, when $k$ has been found and the depth of the secondary minimum is approximately known, it can be made out whether the eclipse at primary minimum is total or annular. In fact, we have :

$$l_1 = 1 - \alpha L_2\ ; \qquad l_2 = 1 - k^2\, \alpha\, L_1$$

or, as $L_1 + L_2 = 1$

$$\alpha = 1 - l_1 + \frac{1-l_2}{k^2}$$

consequently :

$$\alpha_0 = 1 - \lambda_1 + \frac{1-\lambda_2}{k^2} \qquad (4)$$

As has already been remarked on p. 22, the subscript 1 to the quantities $l$ and $\lambda$ refers to the eclipse of the smaller star by the larger ; the subscript 2 to the other eclipse. If at primary minimum the smaller star is eclipsed (case $E_l$) we, therefore, have the equation. :

$$a_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{sec}}{k^2} \qquad (4a)$$

and when at primary minimum the larger star is eclipsed (case $E_s$) :

$$a_0 = 1 - \lambda_{sec} + \frac{1-\lambda_{pr}}{k^2} \qquad (4b)$$

In the following chapters we have always used the equation connecting the depths of the minima in the form (4*a*) or (4*b*).

If the principal eclipse is total or annular we have respectively :

$$\lambda_{sec} = 1 - k^2\ \lambda_{pr} \qquad (4c)$$

or

$$\lambda_{sec} = \frac{1-\lambda_{pr}}{k^2} \qquad (4d)$$

Since $\lambda$ must be less than 1, the first case is always possible. The second case is only possible if $k^2 > 1 - \lambda_{pr}$.

If, then, $k$ has been found, (4*c*) and (4*d*) give the depth of the secundary minimum corresponding to each case, and now it can be decided whether the principal eclipse is total or annular. For either (4*d*) gives $\lambda_{sec} > 1$ and then an annular eclipse should be excluded, or, if (4*d*) gives $\lambda_{sec} < 1$, the depths according to each hypothesis are likely to differ so much [1]) that the

[1]) $\lambda_{sec} - \lambda_{pr} = 1 - \lambda_{pr}\ (1+k^2)$ if the principal eclipse is total.
$\lambda_{sec} - \lambda_{pr} = \frac{1}{k^2} \left\{1 - \lambda_{pr}\ (1+k^2)\right\}$ if the principal eclipse is annular.
The latter hypothesis therefore gives rise to the shallower secondary minimum.

choice is easily made — unless $k$ should be near unity, but then the question is of little importance.

If the primary minimum has a great depth, the eclipse at this minimum can as a rule be directly said to be total, unless the value found for $k$ is large. In dubious cases equations (4$c$) and (4$d$) must decide.

In the case of partial eclipses (cases 3° and 4°) the method just described should be altered. A new unknown quantity is added, viz. $\alpha_0$, the maximum obscuration (which is equal to unity in the case of a total or annular eclipse). RUSSELL finds that in the case of partial eclipses the problem of deriving orbital elements solely from the light-curve of the primary minimum is indeterminate. For any value of $k$, comprised within wide limits, it is possible to find an assumed percentage $\alpha_0$, and hence a set of elements, such that the interval from the middle of eclipse, at which any given magnitude is reached, as calculated from any of these systems of elements, will be the same within a fraction of 1 percent ; in other words, such that nearly the same light-curve will be found. If, however, the depth of the secondary minimum has also been observed, RUSSELL has succeeded in solving this problem ; in this case the results may however be rather doubtful.

For various values $n = \frac{\alpha}{\alpha_0}$ he finds the corresponding values of $\tau$ (and therefore of $\vartheta$) from the light-curve. In a way similar to that followed in the case of a total eclipse, he arrives at a function :

$$\frac{\sin^2 \vartheta\,(n)}{\sin^2 \vartheta\,(\frac{1}{2})} = \frac{\psi\,(k, n\alpha_0) - \psi\,(k, \alpha_0)}{\psi\,(k, \frac{1}{2}\alpha_0) - \psi\,(k, \alpha_0)} = \chi\,(k,\ \alpha_0,\ n) \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The first member contains only known quantities. The second is a function of $k$, $\alpha_0$ and $n$, which may be tabulated for any convenient values. A table for $\chi\,(k,\ \alpha_0,\ \frac{1}{4})$ is constructed, and since it appears that the relations between any pair of the $\chi$-functions, corresponding to different fixed values of $n$, is very nearly linear, we may write in general :

$$\chi\,(k, \alpha_0, n) = w_1\,(n) + w_2\,(n)\,\chi\,(k, \alpha_0, \tfrac{1}{4}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and construct tables for the empirically determined functions $w_1\,(n)$ and $w_2\,(n)$. From (5) and (6) finally :

$$\sin^2 \vartheta\,(n) = w_1\,(n) \sin^2 \vartheta\,(\tfrac{1}{2}) + w_2\,(n) \sin^2 \vartheta\,(\tfrac{1}{4})$$

or, putting $\sin^2 \vartheta\,(\frac{1}{4}) = C$ and $\sin^2 \vartheta\,(\frac{1}{2}) = D$, this becomes

$$\sin^2 \vartheta\,(n) = C\,w_2\,(n) + D\,w_1\,(n) \quad \ldots\ldots\ldots\ldots (7)$$

If the value found for $D = \sin^2 \vartheta\,(\frac{1}{2})$ is preliminarily accepted, equation (7) yields a value of $C = \sin^2 \vartheta\,(\frac{1}{4})$ for any value of $n$. If these values agree sensibly for the upper, lower and middle parts of the light-curve, the mean value of $C$ is taken. If this agreement is not obtained — which is not infrequently the case — an attempt at improvement is made by changing the value of $D$ a little, which is again equivalent to slightly changing the point on the light-curve (i. e. its value for $\tau$, or $\vartheta$) corresponding to $n = \frac{1}{2}$. See further § 7 on this subject.

This being accomplished we get from (5):

$$\chi\,(k,\ \alpha_0,\ \tfrac{1}{4}) = \frac{C}{D} \quad \ldots\ldots\ldots\ldots (8)$$

Unless the secondary minimum has been observed, we can proceed no farther. But if we know the brightness at both minima (4) gives a second relation between $k$ and $\alpha_0$, and values for $k$ and $\alpha_0$ may be derived by means of the table for $\chi\,(k,\ \alpha_0,\ \frac{1}{4})$. The value finally adopted for $\chi\,(k,\ \alpha_0,\ \frac{1}{4})$ by means of (8) may, however, be very uncertain — the separate values of C showing large discrepancies — so that the same may be the case for the values of $k$ and $\alpha_0$ derived from it. Moreover the solution may occasionally be indeterminate, and sometimes two solutions are possible, between which it may be hard to decide.

Finally the elements $r_1$ and $i$ are determined in the same way as in the case of a total eclipse.

Russell has also considered the case of ellipsoidal stars (Lyrids) [1]. The distance between the components of such a system is very small and the period of their revolution most probably equal to that of their axial rotation. Since in consequence of their strong mutual attraction the components have an elongated shape, the longest axis lying in the line of their centers, and since on the other hand the rotation gives them a polar flattening

[1]) Ap.J. 36, 60—67 (1912).

(shortest axis perpendicular to the orbit plane) they will be as a rule ellipsoids of three unequal axes. For the sake of simplification RUSSELL assumes them to be similar and similarly situated.

In the light-curve the ellipsoidal shape of the components is revealed by the continuous light-change between the eclipses, the brightness reaching a maximum value midway. Strictly speaking this is the case with every eclipsing binary. In most cases, however, the brightness between the eclipses may be supposed to be practically constant if the distance between the components is not very small. If, however, there is a distinct deviation from a constant maximum light, we speak of a Lyrid (so-called after the best-known representative, β *Lyrae*, of this group).

The ellipsoidal shape of the components, however, is not the only cause of a continuous light-change between the minima. If the distance of the components is so small, that they are nearly in contact, the light-curve must also on this account present the same characteristics, a constant maximum light being either absent or of so short a duration that it can hardly be observed.

The Lyrid-curves therefore differ from the Algol-curves :

1°) on account of the ellipsoidal shape of the components ;

2°) on account of the very small distance of their centers, the disks being nearly in contact.

Either of these two circumstances will make the light-curve pass continuously from one minimum to the other, without constant maximum light.

The polar flattening cannot be determined from the light-curve, but it may be approximately estimated with the aid of plausible relations between the three axes, based on DARWIN'S studies, when the elongation in the equatorial plane has been found. Therefore in his investigation RUSSELL admits the case of two stars which by their mutual attraction have got the shapes of similar prolate spheroids whose longer axes coincide with the line joining their centers ; for if the dimensions of the ellipsoids in the direction perpendicular to the orbital plane are modified in a constant ratio, the ratio of the eclipsed part to the surfaces of the two disks remain identical. If $L_1$ and $L_2$ represent the maximum values of the light of the two components,

i. e. the values for $\vartheta = 90°$; $l_1$ and $l_2$ the amounts of light which would reach us from each component if there were no eclipse; $d_1$ and $d_2$ the apparent lengths of their major axes at that moment; $a_1$ and $a_2$ the maximum values of these axes, we have

$$\frac{l_1}{L_1} = \frac{l_2}{L_2} = \frac{d_1}{a_1} = (1 - \varepsilon^2 \sin^2 i \cos^2 \vartheta)^{\frac{1}{2}},$$

whence, if $l$ is the actual amount of light received by us at any time :

$$l = l_1 + (1-\alpha)\, l_2 = (1 - \varepsilon^2 \sin^2 i \cos^2 \vartheta)^{\frac{1}{2}} \{L_1 + (1-\alpha)\, L_2\}$$

$$= (1 - z \cos^2 \vartheta)^{\frac{1}{2}} \{L_1 + (1-\alpha)\, L_2\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

When $\varepsilon = 0$ (and therefore $z = 0$) this reduces to the familiar formula for spherical stars. The second factor of (9) is constant ($= 1$) when there is no eclipse ($\alpha = 0$); this means that $z$ may be determined graphically from the light-curve outside of eclipses (near $90°$) by plotting for various points the values of $(1 - l^2)$ against the corresponding values of $\cos^2 \vartheta$. The resulting points will lie on a straight line, whose slope gives the desired value of $z$. When eclipse begins the plotted points fall above this straight line and lie on an ascending curve. This method might seem to fail when the stars are in actual contact because in that case the stars continually eclipse one another more or less, except when $\vartheta = 90°$, so that the curve above described has no rectilinear portion. But the eclipsed surface is very small, varying, as can be easily shown, approximately as $\cos^3 \vartheta$. The tangent to the curve determined by the plotted points at the point for which $\cos \vartheta = 0$ can be drawn, and gives the value of $z$ for this case.

Having found $z$, the light-curve may be „rectified", removing all apparent influence of the ellipsoidal shape of the components by subtracting from the observed magnitudes the computed variation due to the latter cause. We then obtain a light-curve of the ordinary „Algol"-form, with constant light between eclipses, which represents the variations in brightness due to eclipse alone.

For the rest the solution runs parallel to that for spherical stars, only slight changes being necessary in the formulas, since $r_1^2$ is replaced by $d_1^2 = a_1^2\,(1 - z \cos^2 \vartheta)$.

When the orbital elements have been determined, we may express the density of each of the components in terms of the sun's density, putting

$$\rho_1 = 0.01344 \frac{y}{P^2 r_1^3} \quad ; \quad \rho_2 = 0.01344 \frac{1-y}{P^2 r_1^3} \text{ [1]}.$$

The mass of the larger star is represented by $my$ and that of the smaller star by $m(1-y)$, the total mass of the system, expressed in terms of the sun's mass, being $m$. The actual densities cannot be computed unless the ratio of the masses of the two stars, and consequently $y$, is known. As a rule this is not the case. But for a number of visual and spectroscopic binaries the brighter star has proved to have nearly always the greater mass, which, however, does not much exceed that of the smaller star (4 : 1 being the maximum ratio that has hitherto been found).

Therefore in the above formula $y$ is supposed to be $\frac{1}{2}$, i. e. the masses of both components are supposed to be equal. At any rate the *order* of density is found in this way, for which we find

$$\rho = \frac{0.00672}{P^2 r^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

This formula is likely to give too high a density for the faint star and too low for the bright one, but in neither case the error is at all likely to exceed 50 per cent of the computed values, or to be in the opposite sense from that stated.

When the stars are ellipsoidal, our formula obviously becomes $\rho = \frac{0.00672}{P^2 abc}$, where $a$, $b$ and $c$ are the three axes of the ellipsoid.

Finally RUSSELL has considered the effects of an *eccentric orbit* and of *reflexion*. These effects are usually so small as to be detected only by the most refined observations from the brightness between the principal eclipses. We shall not discuss these possibilities [2]), since we have not taken them into account in the examples of chapter IV.

## C. D-HYPOTHESIS.

The hypothesis of uniformly illuminated disks is most probably

[1]) Ap.J. 36, 73 (1912).

[2]) Ap.J. 36, 54—60 and 67—69 (1912).

incorrect, since most stars, like our sun, will show a decrease of brightness towards the limb. RUSSELL assumes it to obey the following law:

$$I = I_0 (1 - x + x \cos \varphi).$$

in which $I$ is the surface brightness and $\varphi$ the angle between the line of sight and the normal to the surface. The coefficient of darkening $x$ has for uniformly illuminated disks (U-hyp.) the value zero and for complete darkening towards the limb (D-hyp.) the value 1. Only these two extreme cases are worked out; in the cases occurring in practice, $x$ is a fraction which is very difficult to estimate, because we must then have a precise determination of the whole light-curve (including the non-eclipse portions as well as the secondary minimum) of either a star with a conspicuous constant phase at principal minimum (the shallower the better), or a star in which the sum of the losses of light at the two minima (after correction for ellipticity of the components) is nearly equal to the whole light of the system.

In the D-hyp. we shall no longer be able to distinguish by inspection of the primary minimum between annular and partial eclipses, because in the former case a constant minimum light is absent. Moreover the relation (4) no longer holds good on account of the dependence of the depth of the minima upon the darkening-coefficient as well as upon $k$ and the area obscured.

First the case of a total eclipse at primary minimum has to be considered. By mechanical quadrature RUSSELL determines values of $\alpha'$ for various values of $p$ and $k$. From this tabulated material curves, represented by equations of the form $\alpha' = f\ (k,p)$ were drawn for fixed values of $k$. Having assigned definite values to $k$, it is possible to invert the $\alpha'$-function just determined and read off from the curves $p$ as a function of $\alpha'$ and $k$. Thus he gets the $p$-table for this hypothesis (Table II at the end of this paper) and with the aid of this table he derives the other tables, the method being for the rest the same as in the U-hyp. [1])

From RUSSELL's closer investigation of systems with total eclipses it appears that, if darkening really exists, the solution made without regarding it (U-hyp.) will lead to a density too low for the large (and usually faint)

[1]) Ap. J. 36, 239 (1912). H. N. RUSSELL and H. SHAPLEY: On darkening at the limb in eclipsing variables, I.

component and too high for the smaller (and usually bright) one. The most important effect is on the computed density of the smaller star, which on the average will be about twice as great in the U-hyp. as in the D-hyp. These densities are computed on the assumption that both components have equal masses. As it is well known that the brighter component of a close double star is regularly the more massive, this component will generally be more dense, and the fainter less dense, than computed on the above assumption. It is probable then, that the computed „uniform" densities very nearly represent the real conditions — the unequal masses being compensated for by a considerable degree of darkening toward the limb.

As to the partial eclipses[1]) it appears that besides equation (4), the empirical linear relations between the $\chi$-functions which fortunately saved the „uniform" partial eclipse problem from a long trial and error process, likewise fail to hold for darkened stars.

The question whether an eclipse is annular or partial can arise only when the larger of the two components is considerably the brighter, and the smaller one is in front of it during the principal minimum, for otherwise the minimum at which total eclipse is possible will be deep enough to make it certain whether or not it shows a constant phase. When this cannot be definitely decided upon, the partial and the annular eclipses have to be treated as one problem.

By using new tables, viz.:

1°) a table giving the value of $p$ for various values of $k$ and $\alpha''$ — this latter quantity may now be greater than unity —;

2°) a table for the function $\psi\ (k, \alpha'')$;

3°) tables for the functions $\chi\ (k, \alpha_0, n)$ for $n = \frac{3}{4}$, $n = \frac{1}{4}$ and $n = 0$; and by using a „trial and error" process, RUSSELL has succeeded in giving solutions for the following cases:

1 $a$) partial eclipse at primary minimum, with larger star in front;

1 $b$) „ „ „ „ „ „ smaller „ „ „ ;

2) annular eclipse at primary minimum.

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1) Ap. J. 36, 385 (1912). H. N. RUSSELL and H. SHAPLEY: On darkening at the limb in eclipsing variables. II.

Sometimes, however, two solutions present themselves, between which there is no choice.

The relation connecting the depths of the minima with $k$ and $\alpha_0$ is now more complicated than in the U-hypothesis (equation (4)) for if the stars are of unequal radii the intensity-curves of primary and secondary minimum are no longer connected by the simple relations $1-l_1 = \alpha L_2$ and $1-l_2 = k^2 \alpha L_1$ ; and one may show a constant minimum light when the other does not. If $\alpha_0'$ (as stated on *p.* 23) denotes the fraction of the light of the smaller star which is lost at the greatest phase of its eclipse behind the larger, RUSSELL calls the fraction of the light of the larger star which is lost at the corresponding phase during the other eclipse $\alpha_0'\ Q\ (k, \alpha_0')$ and gives a table for the new function $Q\ (k, \alpha_0')$ [1]) (Table C at the end of this paper). Further we have as before :

$$1-\lambda_1 = \alpha_0' L_2\,;\quad 1-\lambda_2 = \alpha_0' L_1 Q(k, \alpha_0'),$$

whence, since $L_1 + L_2 = 1$ :

$$\alpha_0' = 1-\lambda_1 + \frac{1-\lambda_2}{Q\ (k, \alpha_0')} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

(For uniform disks the $Q$-function reduces to $k^2$).

In the following chapters we will use this equation in the following forms :

$$\alpha_0' = 1-\lambda_{pr} + \frac{1-\lambda_{sec}}{Q\ (k, \alpha_0')} \text{ (larger star in front at primary minimum; case } E_l') \ldots (11a)$$

or

$$\alpha_0' = 1-\lambda_{sec} + \frac{1-\lambda_{pr}}{Q\ (k, \alpha_0')} \text{ (smaller star in front at primary minimum; case } E_s') \ldots (11b)$$

As has already been stated (p. 22), the loss of light when the large star is eclipsed at primary minimum is expressed in the loss of light at the moment of internal tangency as a unit. At the corresponding moment in the other eclipse the smaller star would be totally hidden, and $\alpha_0' = 1$. Thus, according to the above formulas, we have : $1-\lambda_2 = L_1 Q (k,\ 1)$ and this is the unit to be used in measuring the maximum obscuration $\alpha_0''$. We have therefore the relation :

$$\alpha_0'' Q (k,\ 1) = \alpha_0' Q (k,\ \alpha_0') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

which gives the value of $\alpha_0''$ for each pair of values of $k$ and $\alpha_0'$.

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[1]) Ap.J. 36, 394 (1912).

If we have to deal with ellipsoidal stars, the quantity $Z$ is determined from:[1])

$$l = 1 - Z \cos^2 \vartheta,$$

$$\text{where } Z = \left(\frac{4}{5} \varepsilon^2 + \frac{16}{175} \varepsilon^4 + \ldots\ldots\ldots\right) \sin^2 i,$$

by plotting for various points on the non-eclipse portion of the light-curve ($\vartheta$ near 90°) the values of $1-l$ against the corresponding values of $\cos^2 \vartheta$.

With the aid of the value found for $Z$ the light-curve is „rectified" by subtracting the change in stellar magnitude due to ellipticity from the observed magnitudes. Next the quantity $z = \varepsilon^2 \sin^2 i$, appearing in the formulas, is determined by

$$z = \frac{5}{4} Z - \frac{5}{28} Z^2,$$

and the rectified curve treated in the way already described.

The rectification-factor $Z$ is often (especially if the stars are nearly spherical) almost equal to half the corresponding factor $z$ from the U-hyp.; consequently the factor $z$ to be used in the formulas of the D-hyp. is about $\frac{5}{8}$ of the value of $z$ in the U-hyp.

[1]) Ap.J. 36, 400 (1912).

# CHAPTER III.

## Revised Method of Deriving the Elements of Eclipsing Binaries.

§ 7. *Objections to* Russell's *Method.*

After new magnitudes for the comparison-stars had been obtained, so as to bring them into closer agreement with the step-estimates (see § 4), it was curious to see how much some of the most unmanageable light-curves improved. But there remained some difficulties, which could not be due to erroneous magnitudes of the comparison-stars. Their origin obviously lay in the method employed and it soon became evident, that they chiefly arose from the fact, that the elements — especially the fundamental ratio $k$ — are based on the choice of two fixed points ($a$ and $b$).

1°) The determination of $k$ from points of the light-curve near and between the fixed points $a$ and $b$ appears to be very uncertain (see also p. 26, note 1), small variations in $\tau$ bringing about large changes in $k$. If, for instance, we change the values of $\tau$ belonging to $\alpha = 0.50$ ; $0.70$ ; $0.80$ ; $0.95$ for the curve II (table 12) from $0^d.168^5$ ; $0^d.140$ ; $0^d.124$ and $0^d.087$ into $0^d.168$ ; $0^d.140^5$ ; $0^d.124^5$ and $0^d.086^5$ respectively, the corresponding values of $k$ : 0.20 ; 0.17 ; 0.00 ; 0.11 become 0.12 ; 0.00 ; ? ; 0.05. And if $\tau = 0^d.124$ for the point belonging to $\alpha = 0.80$ is changed into $\tau = 0.123^5$ the corresponding value of $k$ changes from 0.00 into 0.11.

Now, the part of the light-curve between the points corresponding to $\alpha = 0.40$ and $\alpha = 0.95$ is that, which the observations can determine with the greatest precision ; hence we should prefer it in deriving reliable values of $k$. But for the reason mentioned above Russell is obliged to attach less weight to the values determined from it.

Likewise slight changes in the values of $\tau$ for the points $a$ and $b$ may cause great alterations in the values of $k$ especially for the above mentioned points near and between $a$ and $b$. Therefore slightly discrepant values of $\tau$ for the fixed points $a$ and $b$ may be responsible for the above mentioned systematic deviations which the series of values for $k$ often present (see p. 10). These systematic deviations, however, may also be due to the fact that usually the observed light-curve is less reliable between $\alpha = 0.00$ and $\alpha = 0.40$ and again between $\alpha = 0.98$ and $\alpha = 1.00$.

The choice and the position of the fixed points $a$ and $b$ have therefore great influence on the results.

2°) Generally: by slightly modifying the light-curve we may obtain curves, which, though they still fulfil the conditions of § 1 (p. 5), may yield in RUSSELL's treatment of the problem values of $k$ varying to a wide extent.

TABLE 12*a*.

| Curve \ τ | $0^d.000$ | $0^d.060$ | $0^d.080$ | $0^d.100$ | $0^d.120$ | $0^d.140$ | $0^d.160$ | $0^d.180$ | $0^d.200$ | $0^d.220$ | $0^d.240$ | $0^d.256$ | $0^d.260$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | $12^m.24$ | $12^m.23$ | $11^m.90$ | $11^m.40$ | $10^m.95$ | $10^m.47$ | $10^m.07$ | $9^m.83$ | $9^m.60$ | $9^m.46$ | $9^m.37$ | $9^m.32$ | $\{0^d.265$<br>$9^m.30$ |
| II | 12.11 | 12.10 | 11.79 | 11.37$^5$ | 10.95$^5$ | 10.56$^5$ | 10.23 | 9.97 | 9.77 | 9.61 | 9.50 | 9.45 | 9.45 |
| II*a* | 12.11 | 12.10 | 11.77 | 11.36 | 10.96 | 10.56 | 10.22 | 9.96$^5$ | 9.77 | 9.62 | 9.51 | 9.45 | 9.45 |
| II*b* | 12.11 | 12.10 | 11.79 | 11.39 | 10.97$^5$ | 10.56 | 10.22 | 9.97 | 9.78 | 9.62$^5$ | 9.51 | 9.45 | 9.45 |
| II*c* | 12.11 | 12.10 | 11.79 | 11.38 | 10.95 | 10.56 | 10.21 | 9.96 | 9.77 | 9.61 | 9.50 | 9.45 | 9.45 |
| II*d* | 12.11 | 12.10 | 11.76 | 11.36 | 10.96 | 10.58 | 10.24 | 9.98 | 9.78 | 9.62 | 9.50 | 9.45 | 9.45 |
| II*e* | 12.11 | 12.10 | 11.78$^5$ | 11.38$^5$ | 10.98 | 10.58$^5$ | 10.25 | 9.98 | 9.77 | 9.60 | 9.50 | 9.45 | 9.45 |

TABLE 12*b*.

| | α | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | (0.60) | 0.70 | 0.80 | (0.90) | 0.95 | 0.98 | 0.99 | 1.00 | $k_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | τ | $0^d.264$ | $0^d.231$ | $0^d.211$ | $0^d.195$ | $0^d.180$ | $0^d.166^5$ | $0^d.154$ | $0^d.141^5$ | $0^d.127$ | $0^d.107$ | $0^d.091$ | $0^d.077$ | $0^d.070$ | $0^d.060$ | $0^d.25$ |
| | $k$ | 0.42 | 0.40 | 0.36 | 0.39 | 0.25 | 0.15 | | ? | ? | | ? | ? | ? | ? | |
| II | τ | 0.256 | 0.231 | 0.212$^5$ | 0.197$^5$ | 0.182$^5$ | 0.168$^5$ | 0.154$^5$ | 0.140 | 0.124 | 0.102 | 0.087 | 0.074 | 0.068 | 0.058 | 0.25 |
| | $k$ | 0.27 | 0.30 | 0.27 | 0.28 | 0.23 | 0.20 | | 0.17 | 0.00 | | 0.11 | 0.10 | 0.13 | 0.18 | |
| II*a* | τ | 0.256 | 0.232 | 0.212$^5$ | 0.196$^5$ | 0.182 | 0.168 | 0.154 | 0.140 | 0.124 | 0.101$^5$ | 0.086 | 0.073 | 0.067$^5$ | 0.058 | 0.24 |
| | $k$ | 0.29 | 0.32 | 0.30 | 0.27 | 0.24 | 0.20 | | 0.00 | ? | | 0.06 | 0.10 | 0.13 | 0.20 | |
| II*b* | τ | 0.256 | 0.233 | 0.214 | 0.198 | 0.182$^5$ | 0.168 | 0.154 | 0.140$^5$ | 0.124$^5$ | 0.103 | 0.087$^5$ | 0.074 | 0.068 | 0.058 | 0.28 |
| | $k$ | 0.29 | 0.36 | 0.37 | 0.35 | 0.30 | 0.25 | | ? | ? | | 0.00 | 0.00 | 0.02 | 0.09 | |
| II*c* | τ | 0.256 | 0.231 | 0.213 | 0.197 | 0.181$^5$ | 0.167$^5$ | 0.153$^5$ | 0.140 | 0.124 | 0.102 | 0.087 | 0.074 | 0.068 | 0.058 | 0.26 |
| | $k$ | 0.29 | 0.33 | 0.33 | 0.33 | 0.26 | 0.15 | | ? | ? | | 0.07 | 0.05 | 0.09 | 0.14 | |
| II*d* | τ | 0.256 | 0.232 | 0.213 | 0.198 | 0.183 | 0.169 | 0.155$^5$ | 0.141 | 0.124 | 0.102 | 0.085$^5$ | 0.072$^5$ | 0.067 | 0.058 | 0.20 |
| | $k$ | 0.24 | 0.26 | 0.23 | 0.24 | 0.18 | 0.10 | | 0.06 | 0.06 | | 0.00 | 0.05 | 0.13 | 0.22 | |
| II*e* | τ | 0.256 | 0.230 | 0.212 | 0.197 | 0.183 | 0.169$^5$ | 0.156 | 0.142 | 0.125 | 0.103 | 0.087 | 0.074 | 0.068 | 0.058 | 0.14$^5$ |
| | $k$ | 0.24 | 0.25 | 0.22 | 0.19 | 0.15 | 0.10 | | ? | 0.01 | | 0.03 | 0.06 | 0.11 | 0.17 | |

Table 12*a* contains the coordinates of various light-curves of *SW* = *V*23 *Cygni*. Curves I and II are the curves mentioned on p. 11, while II*a*, II*b*, II*c*, II*d* and II*e* have been obtained by slight modifications of curve II. The curves II*a*, II*b*, II*c* and II*d* represent the observations almost as well as curve II, whereas curve II*e* is not quite so satisfactory.

Table 12*b* gives the values of $k$, deduced after RUSSELL's method, for various values of $\alpha$.

Finally for each curve the value of $k$ has been given (last column of Table 12*b*), obtained by a modified method, to be discussed in § 8. The first six curves practically yield the same value for $k$; only curve II*e*, which represents the observations less satisfactorily, gives a somewhat deviating value. The original curve I, from which a value of $k$ could not be derived after RUSSELL's method, now gives about the same value as the other curves. It may be pointed out, however, that the theoretical curve, based on the elements derived from curve I ($k = 0.25$ ; $r_1 = 0.312$ ; $i = 77°42'$) differs somewhat from this observed curve, as is shown in Table 13.

TABLE 13.

| $\alpha$ | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 | 0.95 | 0.98 | 0.99 | 1.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau_o$ | $0^d.264$ | $0^d.231$ | $0^d.211$ | $0^d.195$ | $0^d.180$ | $0^d.166^5$ | $0^d.154$ | $0^d.141^5$ | $0^d.127$ | $0^d.107$ | $0^d.091$ | $0^d.077$ | $0^d.070$ | $0^d.060$ |
| $\tau_c$ | 0.248 | 0.223 | 0.208 | 0.194 | 0.181 | 0.168 | $0.154^5$ | 0.141 | 0.125 | 0.107 | 0.095 | 0.085 | 0.080 | 0.072 |
| O—C | | | | +0.001 | −0.001 | $-0.001^5$ | $-0.000^5$ | $+0.000^5$ | +0.002 | 0.000 | −0.004 | | | |

Since the differences O—C usually are very small for the points from $\alpha = 0.25$ to $\alpha = 0.95$ (from $0^d.00$ to $0^d.01$), the above differences suggest that, by some cause or other, curve I does not give the real course of light-variation for this system.

At the same time we see here again that we ought to be very careful in changing somewhat the value of $\tau$ for points on the light-curve; very small changes having a great influence on the results.

3°) The chief objection to RUSSELL's method lies in the fact, that, in order to get a suitable $k$-series from the quantities $A$ and $B$ (see p. 25), RUSSELL is obliged to shift the points $a$ and $b$ by modifying $\tau$, while the values 0.6 and 0.9 for $\alpha$ are retained. The ,,corrections'' sometimes amount to $0^d.003$. Since this process alters the course of the curve, the neighbouring points should also undergo a change ; but for these RUSSELL retains the original values of $\tau$. Anyway, the curve which had been previously drawn so as to represent the *observations* as exactly as possible is changed, in order to adapt it to the *theory*.

To this we may add that the introduction of slight modifications in the quantities $A$ and $B$ often requires a rather long time and may be worked out with success only when the observed curve is a very good one. If this is not the case, even if there are but slight deviations from the real light-variation, this process cannot be followed at all with good success, or there is a great uncertainty in the value to be adopted for $k$ and in the values of the other elements.

4°) In the case of a partial eclipse similar objections arise. The series of values for $C$, determined with the value of $D = \sin^2 \vartheta\ (\frac{1}{2})$ as derived from the light-curve, is usually very unsatisfactory ; the differences are as a rule great and frequently show a systematic character. As in the case of total eclipses they may of course be caused by a systematic error in the light-curve. These differences in $C$ usually cannot be smoothed over satisfactorily by changing the value of $D$. Here again the objection remains that the light-curve, as it has originally been drawn through the normals, will be vitiated by RUSSELL'S method, since the change in the value of $D$ means an alteration in the value of $\tau$ for the point of the light-curve corresponding to $n = \frac{1}{2}$.

At any rate the function $\chi\ (k, \alpha_0, \frac{1}{4}) = \frac{C}{D}$ remains usually very uncertain. Even more so the values of $k$ and $\alpha_0$, since they may be perceptibly changed by insignificant alterations in the value of $\chi\ (k, \alpha_0, \frac{1}{4})$. This again may result in a very unstable solution in the case of a partial eclipse, even if the value of $\chi\ (k, \alpha_0, \frac{1}{4})$ has been determined fairly well.

In practice it appears that in the case of a partial eclipse it is as a rule impossible to follow the process of varying the value of D in order to get a somewhat reliable series of values for $C$, and therefore to obtain values of $\alpha_0$, $k$, $r_1$ and $i$ which give a theoretical curve agreeing with the observed curve.

The question now arises whether the method can be modified so as to use the whole light-curve for the determination of $k$ and the other elements, instead of two fixed points playing the leading part. We should also satisfy the requirement that nothing shall be altered in the light-curve as it has been drawn after careful considerations, according to the principles mentioned on p. 5. But then it is desirable to use only the steeper parts of the curve, which as a rule have been determined with greater accuracy and which yield more sharply determined values of $k$. An error e. g. in the magnitude of a comparison-star may cause a fairly great error in $\tau$ in the upper part of the light-curve, and may consequently give greatly deviating values of $k$. Moreover this part of the light-curve may be rather uncertain in consequence of the fact that it contains a smaller number of observations per unit of time. Obviously for the part of the light-curve very near the minimum, similar objections hold good. These parts of the light-curve may afterwards point out whether the observations yielded a discordant light-curve. If the light-curve is a very reliable one, they also might give a decision between the U- and the D-hypothesis.

These considerations have led us to suggest the following revised method, in which RUSSELL'S $p$-tables have been retained.

§ 8. *The primary Light-variation shows a constant minimum Light* ($\alpha_0 = 1.00$).

The following cases may now occur:

I. *U-hypothesis*;

1°) case $U_t$: total eclipse at primary minimum;

2°) case $U_a$: annular eclipse at primary minimum;

II. *D-hypothesis*;

case $D_t$: total eclipse at primary minimum.

On page 25 we have found for the apparent distance of the centers of the disks:

$$\delta^2 = \cos^2 i + \sin^2 i \sin^2 \vartheta \quad \text{and} \quad \delta^2 = r_1^2 (1 + kp)^2,$$

whence:

$$\cos^2 i + \sin^2 i \sin^2 \tfrac{2\pi}{P} \tau = r_1^2 (1 + kp)^2,$$

or, putting $\sin^2 \frac{2\pi}{P} \tau = A$:

$$(1 - A) \sin^2 i + r_1^2 + 2pkr_1 + p^2k^2r_1^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

If we take, say, $m$ values of $\alpha$, the corresponding values of $1-l$ are given by $\alpha = \frac{1-l}{1-\lambda}$ and then the corresponding values of $\tau$ by the light-curve; this enables us to compute the quantities $A$. If, as a first approximation, we choose for $k$ a certain value $k_0$ — for further particulars on this choise see p. 43 — the $p$- tables I or II (for U- en D-hyp. respectively)[1]) give the values of $p$ for each value of $\alpha$ and the chosen value of $k$. We thus get $m$ equations of the form (1), which contain three unknown quantities, viz. $k$, $r_1$ and $i$; they may be combined into three groups.

Suppose $m = 15$ and $\alpha = 0{,}25$ ; 0.30 ; 0.35 ;............... 0.95 respectively and let group I contain 6 equations ($\alpha = 0{,}25$............0.50); group II 5 equations ($\alpha = 0.50$............0.75) and group III 4 equations ($\alpha = 0.80$............0.95). Since the curve is steeper for increasing values of $\alpha$, this choice will make each group represent about equal lengths of the light-curve. Moreover less weight is now given to the upper part of the curve than to the lower, steeper part, which furnishes more reliable values of $\tau$.

Taking the means of the equations of each group, we get:

$$\left.\begin{aligned} (1-\overline{A_1}) \sin^2 i + r_1^2 + 2\,\overline{p_1}\, kr_1^2 + \overline{p_1^2}\, k^2 r_1^2 &= 1 \\ (1-\overline{A_2}) \sin^2 i + r_1^2 + 2\,\overline{p_2}\, kr_1^2 + \overline{p_2^2}\, k^2 r_1^2 &= 1 \\ (1-\overline{A_3}) \sin^2 i + r_1^2 + 2\,\overline{p_3}\, kr_1^2 + \overline{p_3^2}\, k^2 r_1^2 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2)$$

from which $k$, $r_1$ and $i$ must be solved. Subtracting the second equation from the first and third respectively:

$$\left.\begin{aligned} (\overline{A_2}-\overline{A_1}) \sin^2 i + 2\,(\overline{p_1}-\overline{p_2})\, kr_1^2 + (\overline{p_1^2}-\overline{p_2^2})\, k^2 r_1^2 &= 0 \\ (\overline{A_2}-\overline{A_3}) \sin^2 i + 2\,(\overline{p_3}-\overline{p_2})\, kr_1^2 + (\overline{p_3^2}-\overline{p_2^2})\, k^2 r_1^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (2a)$$

and regarding these equations as 2 linear homogeneous equations with $\sin^2 i$, $kr_1^2$ and $k^2 r_1^2$ as unknown quantities, we find:

$$k = \frac{k^2 r_1^2}{kr_1^2} = -2\, \frac{(\overline{A_1}-\overline{A_2})\,(\overline{p_3}-\overline{p_2}) - (\overline{A_3}-\overline{A_2})\,(\overline{p_1}-\overline{p_2})}{(\overline{A_1}-\overline{A_2})\,(\overline{p_3^2}-\overline{p_2^2}) - (\overline{A_3^2}-\overline{A_2^2})\,(\overline{p_1^2}-\overline{p_2^2})} \ldots\ldots\ldots (3)$$

The value of $k$ thus found will be regarded as a second approximation, the first being $k_0$. The new values of $p$ will be taken from tables I or II and then a third approximation will be obtained. The process is to be repeated

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1) Taken from RUSSELL: Ap.J. **35**, 333 (1912) and Ap.J. **36**, 243 (1912).

until no further change is found. As a rule it converges remarkably quickly, but we may still shorten it by taking a value of $k$ lying between $k_0$ and $k_1$. Practice has taught that the exact value of $k$ is not far from $\frac{3\,k_1 + k_0}{4}$. It is remarkable that this way of proceeding, though nearly independent of the choice of $k_0$, always leads to good results.

Substituting now the final value of $k$ and the corresponding values of $\overline{p}$ and $\overline{p^2}$, in the first and the third equation of (2):

$$\left.\begin{array}{l}(1-\overline{A_1})\ \sin^2 i + (1 + 2\overline{p_1}\ k + \overline{p_1^2}\ k^2)\ r_1^2 = 1 \\ (1-\overline{A_3})\ \sin^2 i + (1 + 2\overline{p_3}\ k + \overline{p_3^2}\ k^2)\ r_1^2 = 1\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

we get $\sin^2 i$ and $r_1^2$, hence $i$ and $r_1$. It will be clear that the first and the third equation have been chosen because here the coefficients of $\sin^2 i$ and $r_1^2$ are differing most widely. If required, the solution may be facilitated by using a table which gives the coefficients of $r_1^2$ for various values of $k$.

A first approximation of $k$ may be found from the supposition that the eclipse is central. If $t_1$ is the semi-duration of the eclipse and $t_2$ that of the constant minimum-light we get:

$$\frac{\sin \frac{\pi\, t_2}{P}}{\sin \frac{\pi\, t_1}{P}} = \frac{r_1 - r_2}{r_1 + r_2} = \frac{1-k}{1+k},$$

hence

$$k = \frac{\operatorname{tg} \pi \frac{t_1 - t_2}{P}}{\operatorname{tg} \pi \frac{t_1 + t_2}{P}},$$

for which we may also take the somewhat larger value $k = \frac{t_1 - t_2}{t_1 + t_2}$. Approximate values of $t_1$ and $t_2$ may be taken from the observed light-curve. A graphical construction will easily show that if the total eclipse is not central, $k < \frac{t_1 - t_2}{t_1 + t_2}$. The expression for $k$, just found, is therefore an upper limit.

It is to be remarked that even with a fixed initial value of $k$, for instance $k = \frac{1}{2}$, the process described will, by the use of equation (3), lead to a good result.

The process may be accelerated by a table, which gives for various values of $k$ the corresponding values of $\overline{p_1}$, $\overline{p_2}$, $\overline{p_3}$; $\overline{p_1^2}$, $\overline{p_2^2}$, $\overline{p_3^2}$; see for this table at the bottom of tables I and II.

It remains to be decided whether in the U-hyp. the primary minimum is due to a total or to an annular eclipse. This may be done with the aid of the equations of p. 27 :

$$a_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{sec}}{k^2} \quad \ldots\ldots \text{(case } E_t\text{)} \ldots\ldots \quad (5a)$$

and

$$a_0 = 1 - \lambda_{sec} + \frac{1-\lambda_{pr}}{k^2} \quad \ldots\ldots \text{(case } E_s\text{)} \ldots\ldots \quad (5b)$$

which enables us to see which of the two suppositions gives the closest approximation of $a_0$ to unity, and thus to make our choice.

By the above method the elements of the system have been derived in two suppositions ; the U-hyp. and the D-hyp. respectively. These, however, represent two extreme cases, neither of which is likely to occur in reality ; the actual cases lying as a rule between these two extremes. Therefore the question remains how the degree of darkening towards the limb is to be found.

In order to answer this question we might have recourse :

1°) to that part of the light-curve which as yet has not been used in deriving the elements ;

2°) in the case of a total (annular) eclipse, to the depth of the secondary minimum, which as a rule will come out pretty differently according to the hypothesis (U or D) which served as a base in the computation of the elements.

Unfortunately our knowledge of the secondary minima is as yet very small. Our purpose would be best secured by a system with a sharply determined primary and a deep secondary minimum ; the computed depths are then very different in both hypotheses and the observed depth may give some information.

If for a system of this kind, e.g. $V9 = Z$ *Herculis*, the light-curves were determined with great accuracy for both minima, we might also in another way expect to get some insight in the interesting question of darkening toward the limb. For in this case the minimum at which the eclipse is annular cannot have a constant minimum light.

As to the extreme parts of the light-curve, they are as a rule rather inaccurately determined, while the theoretical curves based on the U- and D-hyp. respectively do not differ much for the parts under consideration.

Summarizing we find, that only with the aid of light-curves enjoying the highest accuracy along their whole course, it will be possible to make a tolerably reliable estimate of the degree of darkening.

How then are we to interpolate between the U- and the D-hyp. for intermediate degrees of darkening (e. g. 0.1 ; 0.2 ;. . . . . . . 0.9) ? It is obvious that the values of $p$ can be interpolated directly from the tables I and II. In the way described the value of $k$ is now to be found ; it appears that this value may be interpolated directly between those for the U-hyp. and for the D-hyp. Then we have to determine that degree of darkening, at which the parts mentioned sub 1°) of the theoretical light-curve coincide as well as possible with those of the observed curve.

It appears in practice that we might as well interpolate directly between the coefficients of $r_1^2$, as they have been found in the U- and in the D-hyp. But this method may be a little risky when the degree of darkening toward the limb amounts to about 0.5 ; in this case it would be better to interpolate between the values $\overline{p_1}$, $\overline{p_1}$, $\overline{p_2}$ ; $\overline{p_3^2}$, $\overline{p_2^2}$, $\overline{p_3^2}$, found in the two hypotheses and to calculate the new coefficients of $r_1^2$ with the aid of the interpolated value of $k$.

There is still another case in which we can state something concerning the darkening. It may occur in systems that gave rise to total eclipses that $\sin^2 i$ comes out $> 1$. This case deserves special treatment, as it may lead to an upper limit for the darkening, since it appears that the D-hyp. for such systems gives a greater value for $i$ than the U-hyp. We may distinguish two possibilities :

1°) the value $\sin^2 i > 1$ presents itself in the U-hypothesis.

In this case the D-hyp. need not be considered. We are at once forced to put $\sin^2 i = 1$, so that, according to (2), we get three equations with two unknown quantities, $k$ and $r_1$ ; these equations lead again to the formula (3) for $k$. Next, adding the three equations (2), in which $\sin^2 i = 1$, and substituting the value of $k$ just found, we derive $r_1$.

2°) The value $\sin^2 i > 1$ only results in the D-hypothesis.

This is more likely to occur than the former case. Since, then, a darkening 1.0 (D-hyp.) gives an imaginary value for $i$, whereas a darkening 0.0 (U-hyp.) gives a real solution, we may interpolate between the two hypotheses (in the manner described on p. 45) so as to make $\sin^2 i$ equal to unity. This degree of darkening (D′) will evidently be an upper limit, consistent with the observed light-curve.

In cases of high refinement of the latter, the possibility subsists of interpolating between the D′- and the U-hypothesis, according to the method explained above.

Finally $r_1$ may be computed as sub 1°).

§ 9. *The primary Light-variation shows no constant minimum-Light.*

The following cases may now occur:

I. *U-hypothesis*; the eclipse at primary minimum may be:
 1°) partial, with larger star in front (case $U_{pl}$);
 2°) ,, ,, smaller ,, ,, ,, (case $U_{ps}$);

II. *D-hypothesis*; the eclipse at primary minimum may be:
 1°) partial, with larger star in front (case $D_{pl}$);
 2°) ,, ,, smaller ,, ,, ,, (case $D_{ps}$);
 3°) annular (case $D_a$).

*U-hypothesis.*

According to p. 27 we have:

$$\alpha_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{sec}}{k^2} \text{ ........(case } E_l) \text{ ......................... } (5a)$$

or

$$\alpha_0 = 1 - \lambda_{sec} + \frac{1-\lambda_{pr}}{k^2} \text{ ........(case } E_s) \text{ ......................... } (5b)$$

The depth of the secondary minimum, which was not made use of in the case of total (annular) eclipses, is now supposed to be known.

From equations (5) limits for $\alpha_0$ and $k$ may be easily derived for both cases ($U_{pl}$ and $U_{ps}$). The upper limit for $\alpha_0$ is 1.00 and the lower limit is that which corresponds to $k = 1.00$. Conversely, the upper limit for $k$ is 1.00 and the lower limit is that which corresponds to $\alpha_0 = 1.00$.

We shall consider 15 fractions $n = \frac{\alpha}{\alpha_0}$ of the greatest obscuration $\alpha_0$, viz. $n = 0.25$ ; $0.30$ ;.........$0.95$. For a known or adopted value of $\alpha_0$, these fractions give 15 values of $\alpha$, which allow us to write down 15 equations of the form (1). Reducing these to a system of the form (2) we again find for $k$ the expression (3).

Here again we may start from some initial value of $k$, provided that it lies between the limits just found. Equation (5*a*) or (5*b*) gives the corresponding value of $\alpha_0$ and then formula (3) yields the value of $k$, following from the light-curve. The initial and the final values of $k$ will in general not agree. With the new value of $k$ we proceed in the same way, while this rather rapidly converging process may be shortened as in the case of a total eclipse. In working it out we shall soon be able to discriminate between the cases $U_{pl}$ and $U_{ps}$ ; it appears that if we hit the right case in computing the value of $\alpha_0$ from (5), the resulting value of $k$ will fall somewhere between the two last approximations ; should it fall outside, then the other case is to be taken.

In practice the best way of proceeding is the following : Adopt, between the limits found, a value of $\alpha_0$ and find with formula (3) the corresponding value of $k$ from the light-curve. The equations (5) give the value of $k$ for both cases $U_{pl}$ and $U_{ps}$. The three values of $k$ will in general not agree. Repeat the process with another value of $\alpha_0$. It will soon be clear whether we have to deal with $U_{pl}$ or with $U_{ps}$ ; and only a small number of repetitions will be wanted in order to get a value of $\alpha_0$ for which the light-curve and one of the equations (4) yield the same value of $k$.

In order to accelerate the process we have derived a number of tables I*a*.....I*e* from table I for various values of $\alpha_0$ (0.90......0.50). These tables will give at once the values of $\overline{p_1}$, $\overline{p_2}$, $\overline{p_3}$ ; $\overline{p_1^2}$, $\overline{p_2^2}$, $\overline{p_3^2}$ for different values of $k$. For intermediate values of $\alpha_0$ these quantities may be obtained by interpolation.

The numerical results may be checked by computing $\gamma$ from equation (9*b*) or (10*b*), § 11. The value thus obtained must agree with that yielded by the depth of the minima. For in the U-hyp. the ratio of the losses of light at both minima will be equal to $\gamma$. We may also compute the quantities $L_2 = \frac{1-\lambda_{pr}}{\alpha_0}$ and $L_1 = \frac{1-\lambda_{sec}}{\alpha_0 k^2}$ (case $U_{pl}$) or $L_1 = \frac{1-\lambda_{pr}}{\alpha_0 k^2}$ and $L_2 = \frac{1-\lambda_{sec}}{\alpha_0}$

(case $U_{ps}$) (p. 27) and examine whether the condition $L_1 + L_2 = 1$ has been satisfied.

Finally the elements $r_1$ and $i$ may be found from equations (4) in the same way as for total eclipses, substituting the value found for $k$ in the first and the third equation of (2).

*D-hypothesis.*

The case $D_{pl}$, which we shall first consider, may be treated in the same way as case $U_{pl}$. Instead of (5) we must now use the relations of p. 35 :

$$\alpha'_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{sec}}{Q(k, \alpha'_0)} \ldots\ldots \text{(case } E_l\text{)} \ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

or

$$\alpha'_0 = 1 - \lambda_{sec} + \frac{1-\lambda_{pr}}{Q(k, \alpha'_0)} \ldots\ldots \text{(case } E_s\text{)} \ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

We may start with an initial value of $\alpha'_0$, arbitrarily chosen, but lying between the limits which may be derived from equation (6$a$) and are the same as those given by equation (5$a$) in the case $U_{pl}$. Formula (3) gives the corresponding value of $k$ from the light-curve, after which Table C furnishes $Q(k, \alpha'_0)$ and then (6$a$) a new value of $\alpha'_0$. The latter gives with (3) a second approximation of $k$ etc.

In practice, as in the U-hyp., it is easier to proceed as follows: Adopt a value of $\alpha'_0$. Then equation (6$a$) gives $Q(k, \alpha'_0)$ and Table C the corresponding value of $k$. The light-curve gives, by means of (3), also a value of $k$. These two values of $k$ will in general not agree, but after some trials the value of $\alpha'_0$, for which the required agreement is obtained, will be easily found.

Here again it seemed convenient to accelerate the process by deriving from table II new tables II$a$ .... II$e$, for different values of $\alpha'_0$ (0.90....0.50). They give the values of $\overline{p_1}$, $\overline{p_2}$, $\overline{p_3}$ ; $\overline{p_1^2}$, $\overline{p_2^2}$, $\overline{p_3^2}$ for any value of $k$. For intermediate values of $\alpha'_0$ these quantities may be obtained by interpolation.

Finally we have to discuss the cases $D_{ps}$ and $D_a$. As has been said on p. 22, we now take the loss of light at the moment of internal contact as a unit ; $\alpha'_0$ and $\alpha''_0$ are then connected by the relation of p. 35 :

$$\alpha''_0\, Q(k, 1) = \alpha'_0\, Q(k, \alpha'_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Table III [1]) gives the values of $p$ for every pair of values of $k$ and $\alpha''$. If the eclipse is central, put $\alpha''_0 = 1 + x$; if it is annular, such values of $\alpha''$ as exceed unity may be taken $1 + 0.2$, $1 + 0.4\, x \ldots . 1 + x$.

With an arbitrarily chosen value of $\alpha'_0$ we get the corresponding value of $k$ from (6$b$) and table C, and then equation (7) gives the corresponding value of $\alpha''_0$, from which, by the aid of the light-curve and formula (3) a new value of $k$ may be found. The two values of $k$ will in general not agree. The process must be repeated until the required agreement is obtained.

Here again we have, in order to accelerate the process, derived from table III new tables III$a$..III$e$ for different values of $\alpha''_0$ (0.90....0.50), though the cases $D_{ps}$ and $D_a$ do not occur very frequently.

The numerical results may be checked by computing the quantities $L_2 = \frac{1-\lambda_{pr}}{\alpha'_0}$ and $L = \frac{1-\lambda_{sec}}{\alpha'_0\, \Omega\,(k, \alpha'_0)}$ (case $D_{pl}$) or $L_1 = \frac{1-\lambda_{pr}}{\alpha'_0\, \Omega\,(k, \alpha'_0)}$ and $L_2 = \frac{1-\lambda_{sec}}{\alpha'_0}$ (case $D_{ps}$) and examining whether the condition $L_1 + L_2 = 1$ has been satisfied.

Summarizing, we may treat the D-hyp. in the following way: From equations (6) we learn whether at the primary minimum the smaller or the larger star is eclipsed. In the latter case, in order to know whether the eclipse at primary minimum is partial or annular, we start from $\alpha'_0 = 1.00$ and proceed as described for the cases $D_{pl}$ and $D_a$. A further criterion might be found in the remark on p. 34. Except for this preliminary question, both cases may be treated along the same lines. If in the case $D_a$ the observed light-curve is of high refinement, the points of the theoretical light-curve, found for $\alpha'' = 1 + 0.2\, x$; etc. might again give some information about the degree of darkening toward the limb.

Finally the elements $r_1$ and $i$ will be found by means of equations (4).

## § 10. *Ellipsoidal stars.*

In the case of a Lyrid the light-curve has to be rectified by the factor $z = \varepsilon^2 \sin^2 i$ (see p. 31).

[1]) Taken from Russell: Ap.J. 36, 390 (1912).

Let at any moment $d_1$ and $d_2$ be the apparent major axes of the two stars and $a_1$ and $a_2$ their maximum values, then :

$$d_1^2 = a_1^2 (1 - z \cos^2 \vartheta),$$

and we get :

$$\delta^2 = \cos^2 i + \sin^2 i \sin^2 \vartheta = d_1^2 (1 + kp)^2 = a_1^2 (1 - z \cos^2 \vartheta)(1 + kp)^2.$$

If we put again $\sin^2 \vartheta = A$ and $1 - z \cos^2 \vartheta = 1 - (1 - A) z = B$, we now obtain, instead of (1), equations of the form :

$$(1 - A) \sin^2 i + B \; a_1^2 \; (1 + kp)^2 = 1$$

or

$$\left(\frac{1}{B} - \frac{A}{B}\right) \sin^2 i + a_1^2 + 2\, pk\, a_1^2 + p^2 k^2 a_1^2 = \frac{1}{B} \ldots\ldots\ldots\ldots\ldots (1^*)$$

Taking again, say, 15 of these equations and combining them into three groups, as before, we obtain three equations of the form :

$$\left.\begin{aligned} \alpha_1 \sin^2 i + (1 + 2\, \overline{p_1} k + \overline{p_1^2} k^2)\, a_1^2 &= \beta_1 \\ \alpha_2 \sin^2 i + (1 + 2\, \overline{p_2} k + \overline{p_2^2} k^2)\, a_1^2 &= \beta_2 \\ \alpha_3 \sin^2 i + (1 + 2\, \overline{p_3} k + \overline{p_3^2} k^2)\, a_1^2 &= \beta_3 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2^*)$$

in which

$$\alpha_1 = \left(\frac{1}{B_1} - \frac{A_1}{B_1}\right); \; \beta_1 = \frac{1}{B_1}; \text{ etc.}$$

We may solve $\sin^2 i$ from the second equation and substitute this value in the first and third equations :

$$\{\alpha_2 (1 + 2\, \overline{p_1} k + \overline{p_1^2} k^2) - \alpha_1 (1 + 2\, \overline{p_2} k + \overline{p_2^2} k^2)\}\, a_1^2 = \alpha_2 \beta_1 - \alpha_1 \beta_2$$
$$\{\alpha_2 (1 + 2\, \overline{p_3} k + \overline{p_3^2} k^2) - \alpha_3 (1 + 2\, \overline{p_2} k + \overline{p_2^2} k^2)\}\, a_1^2 = \alpha_2 \beta_3 - \alpha_3 \beta_2$$

Dividing the first of these equations by the second and putting $\frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 \beta_3 - \alpha_3 \beta_2} = \rho$, we get, after some reduction :

$$k^2 + 2\, \frac{\alpha_2 \overline{p_1} - \alpha_1 \overline{p_2} + (\alpha_3 \overline{p_2} - \alpha_2 \overline{p_3})\, \varrho}{\alpha_2 \overline{p_1^2} - \alpha_1 \overline{p_2^2} + (\alpha_3 \overline{p_2^2} - \alpha_2 \overline{p_3^2})\, \varrho}\, k + \frac{\alpha_2 - \alpha_1 + (\alpha_3 - \alpha_2)\, \varrho}{\alpha_2 \overline{p_1^2} - \alpha_1 \overline{p_2^2} + (\alpha_3 \overline{p_2^2} - \alpha_2 \overline{p_3^2})\, \varrho} = 0 \ldots (3^*)$$

If we start with a certain value of $k$ and the corresponding values of $\overline{p}$ and $\overline{p^2}$, equation (3*) gives a better approximation ; the process is repeated as above (p. 41), until the required agreement has been arrived at.

When $k$ has been found, the other elements $\alpha_1$ and $i$ are determined by:

$$\left.\begin{array}{l} \alpha_1 \sin^2 i + (1 + 2\,\bar{p}_1\,k + \bar{p}_1^2\,k^2)\,a_1^2 = \beta_1 \\ \alpha_3 \sin^2 i + (1 + 2\,\bar{p}_3\,k + \bar{p}_3^2\,k^2)\,a_1^2 = \beta_3 \end{array}\right\} \dots\dots\dots\dots\dots\dots (4^*)$$

It is evident, that the method of p. 42 can no longer be applied to the case of ellipsoidal stars, on account of the appearance of the quantities $B$; the second members of the equations (1*) being no longer equal to each other, we cannot derive the peculiar equations ($2a$) nor the simple formula (3).

It is obvious that the general Lyrid-problem, here considered, will be reduced to the ordinary Algol-problem with a constant maximum light, by putting $B = 1$ and consequently $\beta = 1$.

We then have $\rho = \frac{a_2 - a_1}{a_2 - a_3}$ and $\alpha_1 = 1 - A_1$ etc. Substituting the value of $\rho$ in (3*), the numerator of the third term becomes zero and we obtain for $k$:

$$k = -2\,\frac{a_2\,\bar{p}_1 - a_1\,\bar{p}_2 + (a_3\,\bar{p}_2 - a_2\,\bar{p}_3)\,\varrho}{a_2\,\bar{p}_1^2 - a_1\,\bar{p}_2^2 + (a_3\,\bar{p}_2^2 - a_2\,\bar{p}_3^2)\,\varrho}$$

$$= -2\,\frac{(a_2 - a_3)\,(a_2\,\bar{p}_1 - a_1\,\bar{p}_2) + (a_2 - a_1)\,(a_3\,\bar{p}_2 - a_2\,\bar{p}_3)}{(a_2 - a_3)\,(a_2\,\bar{p}_1^2 - a_1\,\bar{p}_2^2) + (a_2 - a_1)\,(a_3\,\bar{p}_2^2 - a_2\,\bar{p}_3^2)}$$

$$= -2\,\frac{(a_1 - a_2)\,(\bar{p}_3 - \bar{p}_2) - (a_3 - a_2)\,(\bar{p}_1 - \bar{p}_2)}{(a_1 - a_2)\,(\bar{p}_3^2 - \bar{p}_2^2) - (a_3 - a_2)\,(\bar{p}_1^2 - \bar{p}_2^2)}$$

$$= -2\,\frac{(A_1 - A_2)\,(\bar{p}_3 - \bar{p}_2) - (A_3 - A_2)\,(\bar{p}_1 - \bar{p}_2)}{(A_1 - A_2)\,(\bar{p}_3^2 - \bar{p}_2^2) - (A_3 - A_2)\,(\bar{p}_1^2 - \bar{p}_2^2)}$$

This is again the formula found on p. 42.

With a high degree of approximation a similar value of $k$ may be derived from equation (3*). Substituting the value of $\rho$, we get:

$$k^2 + 2\,\frac{(a_2\,\beta_3 - a_3\,\beta_2)\,(a_2\,\bar{p}_1 - a_1\,\bar{p}_2) + (a_2\,\beta_1 - a_1\,\beta_2)\,(a_3\,\bar{p}_2 - a_2\,\bar{p}_3)}{(a_2\,\beta_3 - a_3\,\beta_2)\,(a_2\,\bar{p}_1^2 - a_1\,\bar{p}_2^2) + (a_2\,\beta_1 - a_1\,\beta_2)\,(a_3\,\bar{p}_2^2 - a_2 - \bar{p}_3^2)}\,k +$$

$$+ \frac{(a_2 - a_1)\,(a_1\,\beta_3 - a_3\,\beta_1) + (a_3 - a_1)\,(a_2\,\beta_1 - a_1\,\beta_2)}{(a_2\,\beta_3 - a_3\,\beta_2)\,(a_2\,\bar{p}_1^2 - a_1\,\bar{p}_2^2) + (a_2\,\beta_1 - a_1\,\beta_2)\,(a_3\,\bar{p}_2^2 - a_2\,\bar{p}_3^2)} = 0$$

or, after some reduction:

$$k^2 + 2\,\frac{(a_1\,\beta_2 - a_2\,\beta_1)\,(\bar{p}_3 - \bar{p}_2) - (a_3\,\beta_2 - a_2\,\beta_3)\,(\bar{p}_1 - \bar{p}_2) - \bar{p}_2\,\{(a_1 - a_2)\,(\beta_3 - \beta_2) - (a_3 - a_2)\,(\beta_1 - \beta_2)\}}{(a_1\,\beta_2 - a_2\,\beta_1)\,(\bar{p}_3^2 - \bar{p}_2^2) - (a_3\,\beta_2 - a_2\,\beta_3)\,(\bar{p}_1^2 - \bar{p}_2^2) - \bar{p}_2^2\,\{(a_1 - a_2)\,(\beta_3 - \beta_2) - (a_3 - a_2)\,(\beta_1 - \beta_2)\}}\,k -$$

$$- \frac{(a_1 - a_2)\,(\beta_3 - \beta_2) - (a_3 - a_2)\,(\beta_1 - \beta_2)}{(a_1\,\beta_2 - a_2\,\beta_1)\,(\bar{p}_3^2 - \bar{p}_2^2) - (a_3\,\beta_2 - a_2\,\beta_3)\,(\bar{p}_1^2 - \bar{p}_2^2) - \bar{p}_2^2\{(a_1 - a_2)\,(\beta_3 - \beta_2) - (a_3 - a_2)\,(\beta_1 - \beta_2)\}} = 0 \quad (3^{**})$$

If $\beta_1 = \beta_2 = \beta_3$ this equation reduces to the equation, which yields the value of $k$ given by (3) of p. 42.

If we put

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 = R_1 \; ; \; \alpha_3 \beta_2 - \alpha_2 \beta_3 = R_2 \; ; \; (\alpha_1 - \alpha_2)(\beta_3 - \beta_2) - (\alpha_3 - \alpha_2)(\beta_1 - \beta_2) = R_3,$$

in which the quantities $R$ are constants, easily obtainable, equation (3**) becomes :

$$k^2 + 2 \frac{R_1 (\overline{p}_3 - \overline{p}_2) - R_2 (\overline{p}_1 - \overline{p}_2) - R_3 \overline{p}_2}{R_1 (p_3^2 - p_2^2) - R_2 (p_1^2 - p_2^2) - R_3 p_1^2} k - \frac{R_3}{R_1 (p_3^2 - p_2^2) - R_2 (p_1^2 - p_2^2) - R_3 p_2^2} = 0. \quad (5^*)$$

Finally :

$$R_1 = \alpha_1 \beta_2 - \alpha_2 \beta_1 = \frac{1}{\overline{B}_2}\left(\frac{1}{\overline{B}_1} - \frac{\overline{A}_1}{\overline{B}_1}\right) - \frac{1}{\overline{B}_1}\left(\frac{1}{\overline{B}_2} - \frac{\overline{A}_2}{\overline{B}_2}\right) = \frac{1}{\overline{B}_1} \cdot \frac{\overline{A}_2}{\overline{B}_2} - \frac{1}{\overline{B}_2} \cdot \frac{\overline{A}_1}{\overline{B}_1}$$

With a high degree of approximation we may write for this value :

$$R_1 = \frac{\overline{A}_2 - \overline{A}_1}{\overline{B}_1 \overline{B}_2}. \text{ In the same way : } R_2 = \frac{\overline{A}_2 - \overline{A}_3}{\overline{B}_2 \overline{B}_3},$$

$$\text{and } R_3 = \frac{\overline{A}_3 - \overline{A}_1}{\overline{B}_1 \overline{B}_3} + \frac{\overline{A}_1 - \overline{A}_2}{\overline{B}_1 \overline{B}_2} + \frac{\overline{A}_2 - \overline{A}_3}{\overline{B}_2 \overline{B}_3}.$$

Substituting these values in (5*), we finally find :

$$k^2 + 2 \frac{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(\overline{p}_3 - \overline{p}_2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(\overline{p}_1 - \overline{p}_2) - \{\overline{B}_2 (\overline{A}_1 - \overline{A}_3) + \overline{B}_3 (\overline{A}_2 - \overline{A}_1) + \overline{B}_1 (\overline{A}_3 - \overline{A}_2)\} \overline{p}_2}{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(p_3^2 - p_2^2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(p_1^2 - p_2^2) - \{\overline{B}_2 (\overline{A}_1 - \overline{A}_2) + \overline{B}_3 (\overline{A}_2 - \overline{A}_1) + \overline{B}_1 (\overline{A}_3 - \overline{A}_2)\} p_2^2} k -$$

$$- \frac{\overline{B}_2 (\overline{A}_1 - \overline{A}_3) + \overline{B}_3 (\overline{A}_2 - \overline{A}_1) + \overline{B}_1 (\overline{A}_3 - \overline{A}_2)}{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(p_3^2 - p_2^2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(p_1^2 - p_2^2) - \{\overline{B}_2 (\overline{A}_1 - \overline{A}_3) + \overline{B}_3 (\overline{A}_2 - \overline{A}_1) + \overline{B}_1 (\overline{A}_3 - \overline{A}_2)\} p_2^2} = 0. \quad (6^*)$$

For an Algol-system $\overline{B}_1 = \overline{B}_2 = \overline{B}_3 = 1$ and then this equation reduces to the equation, which yields the value of $k$ given by (3) of p. 42.

Should it appear that the expression

$$\overline{B}_2 (\overline{A}_1 - \overline{A}_3) + \overline{B}_3 (\overline{A}_2 - \overline{A}_1) + \overline{B}_1 (\overline{A}_3 - \overline{A}_2)$$

is very small — which may be examined before and always seems to be the case — equation (6*) may be written in the same form as obtained for stars with constant maximum light, viz. :

$$k^2 + 2 \frac{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(\overline{p}_3 - \overline{p}_2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(\overline{p}_1 - \overline{p}_2)}{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(p_3^2 - p_2^2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(p_1^2 - p_2^2)} k = 0$$

or :

$$k = -2 \frac{\overline{B}_1 (\overline{A}_1 - \overline{A}_2)(\overline{p}_3 - \overline{p}_2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(\overline{p}_1 - \overline{p}_2)}{\overline{B}_3 (\overline{A}_1 - \overline{A}_2)(p_3^2 - p_2^2) - \overline{B}_1 (\overline{A}_3 - \overline{A}_2)(p_1^2 - p_2^2)} \quad \ldots\ldots\ldots\ldots\ldots (7^*)$$

In the same way equation (5*) may be approximated to

$$k^2 + 2\,\frac{R_1\,(p_3-p_2) - R_2\,(p_1-p_2)}{R_1\,(p_3{}^2-p_2{}^2) - R_2\,(p_1{}^2-p_2{}^2)}\quad k = 0$$

or:

$$k = -2\,\frac{R_1\,(p_3-p_2) - R_2\,(p_1-p_2)}{R_1\,(p_3{}^2-p_2{}^2) - R_2\,(p_1{}^2-p_2{}^2)}\;\ldots\ldots\ldots\ldots\ldots\ldots (8^*)$$

Since the quantities $\alpha$ and $\beta$ are required when we want to get $i$ and $a$ from equation (4*), the best way is to use (8*) instead of (7*).

The approximated values from (7*) and (8*) appear to be virtually the same as those yielded by the rigorous equations (5*) and (6*).

§ 11. *Determination of theoretical Light-curves.*

The elements $i$, $r_1$, $k$ and $\gamma$ are supposed to be known. From the relation:

$$\cos^2 i + \sin^2 i \sin^2 \frac{2\pi}{P}\tau = r_1{}^2\,(1 + k\,p)^2$$

we get:

$$\sin^2 \frac{2\pi}{P}\tau = \frac{r_1{}^2\,(1 + k\,p)^2 - \cos^2 i}{\sin^2 i}\;\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Let $l_0 = 1$ be the maximum light of the system, $l$ the light at a certain moment and $\lambda$ the light at minimum; let $S$ and $s$ be the surfaces of the disks of the larger and of the smaller star respectively; $\alpha$ the eclipsed part of the disk of the smaller star at a certain moment and $\alpha_0$ the eclipsed part at minimum.

Then we have for the case $E_l$:

$$\frac{l}{l_0} = \frac{S + s\gamma - \alpha s \gamma}{S + s\gamma}$$

or, since $\frac{s}{S} = k^2$ and $l_0 = 1$:

$$l = 1 - \frac{\alpha k^2 \gamma}{1 + k^2 \gamma}\;\ldots\ldots\ldots\ldots\ldots\ldots (9a)$$

so that:

$$\lambda_{pr} = 1 - \frac{\alpha_0 k^2 \gamma}{1 + k^2 \gamma}\;\ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

And for the case $E_s$:

$$\frac{l}{l_0} = \frac{S + s\gamma - \alpha k^2 S}{S + s\gamma}$$

or

$$l = 1 - \frac{\alpha k^2}{1 + k^2 \gamma}\;\ldots\ldots\ldots\ldots\ldots\ldots (10a)$$

so that:

$$\lambda_{pr} = 1 - \frac{\alpha_0 k^2}{1 + k^2 \gamma}\;\ldots\ldots\ldots\ldots\ldots\ldots (10b)$$

The formulae (9$b$) and (10$b$) appear to be the same as formulae (4$a$) and (4$b$) on p. 27. From (9$b$) it follows: $\alpha_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{pr}}{k^2\gamma}$, or, since in that case $\gamma = \frac{1-\lambda_{pr}}{1-\lambda_{sec}}$, we get: $\alpha_0 = 1 - \lambda_{pr} + \frac{1-\lambda_{sec}}{k^2}$.

And in the same way from (10$b$): $\alpha_0 = \frac{1-\lambda_{pr}}{k^2} + (1 - \lambda_{pr})\,\gamma$ and, since now $\gamma = \frac{1-\lambda_{sec}}{1-\lambda_{pr}}$, we find: $\alpha_0 = 1 - \lambda_{sec} + \frac{1-\lambda_{pr}}{k^2}$.

When $\gamma > 1$ we have the cases $U_t$ ($D_t$) or $U_{pl}$ ($D_{pl}$) at primary minimum.
When $\gamma < 1$ ,, ,, ,, ,, $U_a$ ($D_a$) or $U_{ps}$ ($D_{ps}$) ,, ,, ,,
When $r_2 + \cos i < r_1$ the eclipse is partial.

*a) Cases $U_t$ and $U_a$.*

Here $\alpha_0$ is equal to unity.

In the case $U_t$ ($\gamma > 1$), equation (9$b$) gives: $\lambda_{pr} = \frac{1}{1+k^2\gamma}$.

For different values of $\alpha$ Table I yields the corresponding values of $p$, after which equation (8) gives the corresponding values of $\tau$. From equation (9$a$) we derive the loss of light of the system, which is readily translated into a difference of stellar magnitude by means of Table A. The light-curve can now be plotted.

In the case $U_a$ ($\gamma < 1$), equation (10$b$) gives: $\lambda_{pr} = 1 - \frac{k^2}{1+k^2\gamma}$.

For the rest the procedure is as described in case $U_t$, equation (9$a$) now being replaced by equation (10$a$).

*b) Cases $U_{pl}$ and $U_{ps}$.*

In the case $U_{pl}$ ($\gamma > 1$), the relation

$$\delta^2 = \cos^2 i + \sin^2 i \sin^2 \vartheta$$

shows that at the middle of the eclipse $\delta = UU_1 = \cos i$ (Fig. 4 p. 24). At that moment is, according to p. 24:

$$\alpha_0 = \frac{\frac{\beta}{180}\,\pi k^2 + \frac{\varepsilon}{180}\,\pi - \frac{1}{2} k^2 \sin 2\beta - \frac{1}{2} \sin 2\varepsilon}{\pi k^2} \quad \ldots\ldots\ldots\ldots (11)$$

where $\cos\beta = \frac{r_2^2 + \cos^2 i - r_1^2}{2\, r_2 \cos i}$ and $\cos\varepsilon = \frac{r_1^2 + \cos^2 i - r_2^2}{2\, r_1 \cos i}$.

Further, according to (9$b$), the light at the middle of the eclipse is: $\lambda_{pr} = 1 - \frac{\alpha_0 k^2 \gamma}{1 + k^2 \gamma}$.

The maximum obscuration $\alpha_0$ is found by means of (11). The values of $p$, corresponding to the different fractions $n$ of the maximum obscuration $\alpha_0$, are easily derived from Table I. E. g. for $\alpha_0 = 0.80$ and $n = 0.40$ we enter that table for the argument $\alpha = 0.32$. Equation (8) gives now the corresponding values of $\tau$. The corresponding losses of light, expressed in magnitudes, are derived by means of (9$a$) and then the light-curve can be plotted.

In the case $U_{ps}$ ($\gamma < 1$) we have, according to (10$b$): $\lambda_{pr} = 1 - \frac{\alpha_0 k^2}{1 + k^2 \gamma}$.

For the rest we proceed as in case $U_{pl}$, equation (10$a$) taking the place of (9$a$).

*c*) *Cases* $D_t$, $D_{pl}$ ; $D_{ps}$ *and* $D_a$.

In these cases, instead of $\alpha$ ($\alpha_0$), we use the quantities $\alpha'$ ($\alpha'_0$) or $\alpha''$ ($\alpha''_0$) i. e. the losses of light expressed in the loss of light at the moment of internal contact (p. 22/23). In case $D_t$ $\alpha'_0 = 1$. In the cases $D_{pl}$ and $D_{ps}$ we may first compute, by means of (11), the eclipsed part $\alpha_0$ of the smaller star and then by the use of Tables I and II — respectively I and III — the corresponding values of $\alpha'_0$ and $\alpha''_0$. For the rest the procedure is the same as in the U-hyp., provided that the values of $p$ in (8) are derived either from Table II (case $D_t$), or from Tables II$a$....II$e$ (case $D_{pl}$), or from Tables III$a$....III$e$ (case $D_{ps}$).

Since, finally, in case $D_a$ we have $\alpha'' = 1$ at the moment of internal contact, we may proceed as in case $U_a$, provided that for $p$ in (8) we use the values from Table III. We now may also use values of $\alpha'' > 1$, until (8) yields for $\tau$ the value 0. Then $\alpha''_0$ has also been found.

*d*) In the foregoing pages we have described how, in general, theoretical curves of eclipsing binaries may be computed. As a rule, however, the elements $k$, $r_1$ and $i$ have been derived from an observed light-curve; we often want to compute the theoretical light-curve with these elements, both to compare it with the observed curve and to check the computations. In this case the process described above may be shortened, as now the magnitudes corresponding to each $\alpha$ or $n$ are available; in fact they have already been determined and used in the course of the computation of the elements and we have only to compute the corresponding values of $\tau$ by means of (8). In other words : the quantity $\gamma$, which determines the amount of light at minimum, is not taken into consideration, since the range of light-varation is given by the observed curve and is used for the derivation of the elements.

In the case of ellipsoidal stars the values of $\tau$ are computed by means of the relation. :

$$\cos^2 i + \sin^2 i \sin^2 \vartheta = a_1^2 (1 - z \cos^2 \vartheta)\ (1 + kp)^2,$$

whence

$$\cos^2 \vartheta = \cos^2 \frac{2\pi}{P}\tau = \frac{1 - a_1^2\ (1 + kp)^2}{\sin^2 i - a_1^2\ z\ (1 + kp)^2} \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Between the points $\alpha\ (n) = 0.25$ and $\alpha\ (n) = 0.95$ the theoretical light-curve will sensibly coincide with the observed curve, provided the latter is tolerably accurately observed. This, however, will as a rule not occur with the extreme parts. These differences and the consequences to which they may lead, have been previously discussed (p. 44).

We may conclude this paragraph by drawing attention to a peculiarity which we have met with while investigating the system *V* 12 = *RT Persei* (See § 15). The light-curve, as drawn through the normals, did not show a constant minimum light ; so far as it was used for the determination of the elements, it led to an eclipse, either just total (U-hyp.) or nearly so (D-hyp.). But the theoretical light-curve began to systemetically deviate from the observed curve after the point for which $n = 0.90$ ; it showed a stationary minimum of short duration. Since the plotted observations did not contradict this eventuality, the lower part of the light-curve was slightly altered and the elements of the system were again derived. They naturally differed but very little from those found before and yielded a theoretical light-curve which showed a perfect agreement with the observed curve.

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## CHAPTER IV.

### Applications.

§ 12. *General Remarks.*

We shall now apply the preceding theory to a small number of observed light-curves.

An explanation of each of the columns in the following Tables 14; 15; 16; 17, prefixed by the heading of the column, is first given:

$\alpha$, or in the case of a partial eclipse $n$: the fraction of the greatest loss of light;

$1-l$: the corresponding loss of light, computed from the loss of light at mid-eclipse;

$m$: the corresponding stellar magnitude, derived from the data in the preceding column by means of Table A;

$\tau$; $\frac{2\pi}{P}\tau$; Degr.: the corresponding phase, in days, radians and degrees respectively;

$A$: the quantities giving $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ which appear in the formulae (3) and (4), from which the elements $k$, $r_1$ and $i$ are computed as explained in Chapter III.

Finally the theoretical light-curve has been derived:

$\tau_c$: the corresponding phases of the theoretical light-curve;

O—C: differences between the observed and the computed curve.

The preceding theory was based upon the light-ratio 2.512. All the magnitudes of the comparison-stars used in the derivation of the light-curve were either directly taken from the HP or reduced to the Harvard-scale. Of late years, however, the exactness of this ratio has been doubted. According to investigations of Nijland[1]) there exists a striking difference

[1]) A.N. **205**, 233 (1917): Uber die Sehgrenze des Utrechter Zehnzöllers und die photometrischen Skalen von E. C. Pickering und J. A. Parkhurst.

See also: Hemel en Dampkring **14**, 65 (1916).

between the photometric magnitudes of faint stars as given by E. C. PICKERING and J. A. PARKHURST, though both assert having used the ratio 2.512. When PICKERING is right, PARKHURST'S ratio must — according to NIJLAND — be 2.05 ; on the other hand if PARKHURST has used the ratio 2.512, the Harvard scale is based on the number 2.94.

The question is most important since the results of many investigations in stellar astronomy depend on the exactness of PICKERING'S magnitudes of faint stars. VAN DER BILT made some investigations with the polarizing photometer of the Utrecht Observatory, which invariably seemed to point to the exactness of PARKHURST'S scale. Later, at the suggestion of KAPTEYN and SEARES, a wire-gauze screen was placed before the objective, when the brighter of two stars was measured. The obscuration caused by the screen was accurately known and applied to the small difference in brightness measured with the photometer ; the light of the faint stars now agreed fairly well with PICKERING'S values. An extensive investigation following both methods and a detailed knowledge, for a large number of individual cases, about the way in which the magnitudes of faint stars have been obtained at Harvard, seem necessary before a definitive judgment can be obtained[1]).

Since, therefore, a decision in this question is impossible for the present, we have decided to use both scales in deriving the intensity-curve from the magnitude-curve. If we should have to adopt PARKHURST'S scale, we might keep the light-curves already obtained, introducing now the ratio 2.05 (Table B).

In the following examples, therefore, the elements will be determined according to four suppositions successively :

A Light-ratio 2.512.

*a*) U-hypothesis ;

*b*) D-hypothesis.

B Light-ratio 2.05.

*a*) U-hypothesis ;

*b*) D-hypothesis.

1) J. V. D. BILT: Note on the photometric scales of PICKERING and PARKHURST. B. A. N. 30, 167 (1922).

If we had a well-determined light-curve with a deep and sharply determined secondary minimum at our disposal, such a curve might give some valuable information as to the controversy between the two light-scales. For, according to formulae 5 (p. 46)

$$\alpha_0 = 1 - \lambda_{pr} + \frac{1 - \lambda_{sec}}{k^2}$$

or

$$\alpha_0 = 1 - \lambda_{sec} + \frac{1 - \lambda_{pr}}{k^2}$$

The combined loss of light at the middle of the two minima has unity for its upper limit ; should the variable be a Lyrid, the light-curve must first be „rectified", which lessens the depths of both minima. According to Tables A and. B the above limit will sooner be reached with PICKERING'S ratio than with PARKHURST'S.

If $k < 1$, the combined loss of light is less than unity ; the limit 1 will only be reached in a central eclipse of equal stars ($k = 1$). If, moreover, the two components should have the same surface-brightness, the loss of light at the middle of the two minima is $1 - \lambda_{pr} = 1 - \lambda_{sec} = 0.5$. This value corresponds to a depth of $0^m.75$ in the case of PICKERING (Table A), but of $0^m.96^5$ (Table B) in the case of PARKHURST. If the eclipse is not central, the two equal minima are shallower. Should we, therefore, meet with a light-curve which shows a primary and a secondary minimum of nearly the same depth which (if necessary after a correction for ellipsoidal form of the components) exceed $0^m.75$, this fact would support the validity of PARKHURST'S scale. It should be borne in mind, however, that if the minima are of equal depth, we ought to make sure first whether they are really primary and secondary and not both primary, the secondary minimum not being observable. For this purpose we may have recourse to spectroscopic observations. In the first case the period is twice as long as in the second. If the light-curve has one pronounced maximum midway between the minima the second supposition must be rejected.

A light-curve fulfilling the crucial conditions has not presented itself as yet. Various light-curves have been found to have equal or almost equal minima — for instance *V* 4 = *U Ophiuchi*, *V* 16 = *RX Herculis* ; *V* 19 = *SS Carinae* ; *V* 31 = *TX Herculis* ; *V* 11 = *RS Scuti* — but the amplitudes are always less than $0^m.75$, with the possible exception of *V* 11 = *RS Scuti*. It is true that so far as a rectification of the curve has been applied, this more or less uncertain process may be as our judgment to a certain degree.

*V* 11 = *RS Scuti* and *V* 12 = *RT Lacertae* are systems, the accurate determination of whose light-curves would be, in this connection, of the highest interest. SHAPLEY has tried to derive elements for these stars [1]), based upon observations of *V* 11 = *RS Scuti* by ICHINOHE and of *V* 12 = *RT Lacertae* by LUIZET and ENEBO. But these observations are not accurate enough, especially in the case of *V* 11 = *RS Scuti*, to yield a reliable light-curve. For this star SHAPLEY, from observations at maximum light, decides upon an ellipsoidal form of the components. Though such a form really appears to exist, it probably has not such a pronounced character as SHAPLEY presumes. Removing the influence of this ellipsoidal form on the brightness, he obtains a ,,rectified'' curve with minima $0^m.75$ deep. According to PICKERING'S scale this is at the very limit of possibility.

As to *V* 12 = *RT Lacertae,* observations of LUIZET and ENEBO make it highly probable that the brightness remains constant between the eclipses. The depths of the two minima are found to be $1^m.06$ and $0^m.61$, so that according to Table A the loss of light at these minima reaches the values 0.6233 and 0.4298 respectively, their sum exceeding therefore unity. The introduction of an ellipsoidal form of the components proved necessary to allow of a solution of the problem ; though, as we pointed out before, this form is hardly compatible with the light-curve at maximum[2]). In PARKHURST'S scale, however, the total loss of light is only $0^m.89$.

Tables A and B show that the combined loss of light at both minima comes out greater in PICKERING'S scale than in PARKHURST'S. The quantity $\alpha_0$, therefore, will sooner attain the value 1.00 in the first case than in the second, so that *k* must lie between narrower limits (see p. 46) when we use PICKERING'S ratio. Hence we should expect that the possibility of

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[1]) Contr. from the Princeton Un. Obs. **3**, (1915) ; 86, 88, 101, 104, 157, 171.

[2]) NIJLAND'S observations, A.N. **211**, 357 (1920), however, showed a very pronounced Lyrid-character, the depths of the minima being $1^m.4$ and $1^m.9$, respectively. There remains, it is true, some doubt as to the constancy of the brightest of the comparison-stars during the years 1916—1921 ; this, however, cannot have affected the shape of the light-curve to any appreciable amount.

deriving elements for the system would be greater when we use PARKHURST'S scale. Nevertheless the examples which will be treated below strongly point to the superiority of PICKERING'S scale.

PARKHURST'S scale sometimes yields no set of elements at all or no satisfactory one. See *V* 9 = *Z Herculis* (§ 14); *V* 48 = *WZ Cygni* (§ 16) and *V* 12 = *RT Persei* (§ 15). It should be borne in mind, however, that a change in the light-curve or in the range of the light-variation resulting from new observations, may materially influence the results.

Such systems as present a total or annular eclipse at primary minimum and an appreciable and reliable depth of the secondary minimum, may be regarded as suitable to investigate the question. For the derivation of the elements the depth of this minimum is not wanted and its theoretical amount may be computed afterwards by means of the value found for $k$, and compared with the observed value. From the formula for $\alpha_0$ and the Tables A and B it follows, that the computed depth is much greater when based on PARKHURST'S scale than on PICKERING'S. The observed depth, therefore, may also be used as a criterion. Such variables are for instance *V* 9 = *Z Herculis* (§ 14); *V* 9 = *Z Vulpeculae*; *V* 27 = *TT Aurigae*; *V* 6 = *W Crucis*.

As to the following examples, we have always assumed the orbit to be circular — the observations do not give any indication of ellipticity — and likewise we have assumed the brightness between the eclipses to be constant; except for the Lyrid *V* 48 = *WZ Cygni*, where this is obviously not the case.

The densities have been expressed in terms of the solar density and are based on the supposition of equal masses of the components of the system.

For every variable the coordinates of the comparison-stars have been given, besides their magnitudes taken from the HP, or reduced to it. The magnitudes printed in italics have been derived from the limits of vision of the instrument used.

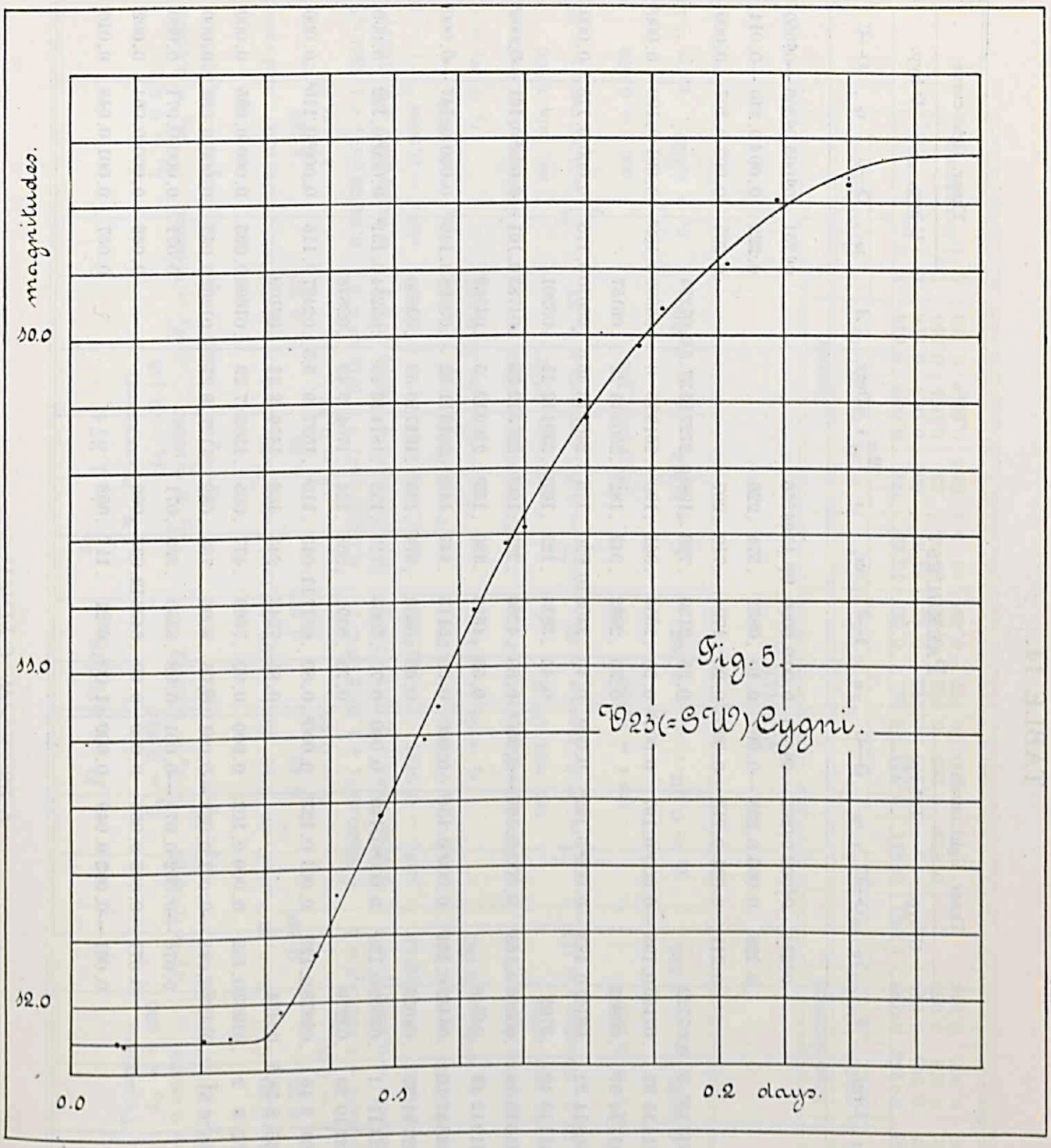
The magnitudes in the column headed H', used in deriving the light-curve, are those found with the step-estimates after the method of § 4.

§ 13. V 23 = SW CYGNI.

| Comparison-stars (1900) | | H.A.74 | H′ | Normals | | | | Light-curve Fig. 5. | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\delta$ | | | | | | | | |
| $20^h3^m\ 5^5$ | $+45°59'.5$ | $8^m.91$ | $8^m.80$ | $-0^d.240$ | $9^m.53^5$ | $+0^d.218$ | $9^m.58^5$ | $0^d.000$ | $12^m.11$ |
| 5 1 | 45 58.4 | 9.39 | 9.64 | 0.203 | 9.78 | 0.183 | 9.92 | 0.058 | 12.11 |
| 3 39 | 46 2.7 | 9.92 | 10.18 | 0.176 | 10.03 | $0.160^5$ | 10.25 | 0.060 | 12.10 |
| 3 31 | 46 3.2 | 10.42 | 10.72 | 0.158 | 10.20 | $0.140^5$ | 10.58 | 0.080 | 11.79 |
| 2 53 | 45 58.8 | *11.18* | 10.96 | 0.136 | 10.62 | 0.125 | 10.83 | 0.100 | 11.38 |
| 3 30 | 46 1.3 | 11.80 | 11.69 | 0.114 | 11.12 | $0.109^5$ | 11.22 | 0.120 | 10.95 |
| 4 21 | 45 57.8 | — | 12.46 | 0.097 | 11.44 | $0.097^5$ | 11.43 | 0.140 | 10.56 |
| 4 28 | 45 58.9 | *13.79* | 13.40 | 0.076 | 11.86 | 0.082 | $11.68^5$ | 0.160 | 10.23 |
| | | | | 0.049 | 12.10 | $0.064^5$ | $12.02^5$ | 0.180 | 9.97 |
| | | | | 0.016 | 12.12 | 0.041 | $12.10^5$ | 0.200 | 9.77 |
| | | | | | | 0.014 | 12.11 | 0.220 | 9.61 |
| | | | | | | | | 0.240 | 9.50 |
| | | | | | | | | 0.256 | 9.45 |
| | | | | | | | | 0.260 | 9.45 |

Maximum $9^m.45$. Primary minimum $12^m.11$. Secondary minimum $9^m.51$.
Depth $2^m.66$. Depth $0^m.06$.

Semi-duration of eclipse : $t_1 = 0^d.260$ ; of constant minimum $t_2 = 0^d.060$. Upper limit of $k$ about $\frac{t_1 - t_2}{t_2 + t_2} = 0.62$. Hence it follows immediately (see p. 44) that the eclipse at primary minimum is total.


magnitudes.
10.0
11.0
12.0
0.0
0.1
0.2 days.
Fig. 5.
V23 (= SW) Cygni.

# TABLE 14.

| Pickering | | | | | | | Theor. light-curve | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | U-hyp. | | D-hyp. | |
| $a$ | $1-l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | $\tau_c$ | O—C | $\tau_c$ | O—C |
| 0.00 | 0.0000 | $9^m.45$ | $0^d.256$ | | | | $0^d.254$ | $0^d.002$ | $0^d.262$ | $-0^d.006$ |
| 0.10 | .0914 | .544 | .231 | | | | 0.228 | 0.003 | 0.233 | —0.002 |
| 0.20 | .1827 | .669 | .212$^5$ | | | | 0.211$^5$ | 0.001 | 0.213 | —0.000$^5$ |
| 0.25 | .2284 | .731 | .205 | 0.2817 | 16° 8′ | 0.07722 | | | | |
| 0.30 | .2741 | .798 | .197$^5$ | .2714 | 15 33 | .07186 | 0.198 | —0.000$^5$ | 0.197 | 0.000$^5$ |
| 0.35 | .3198 | .868 | .190 | .2611 | 14 57$^5$ | .06663 | | | | |
| 0.40 | .3655 | .944 | .182$^5$ | .2508 | 14 22 | .06157 | 0.183 | —0.000$^5$ | 0.182$^5$ | 0.000 |
| 0.45 | .4112 | 10.025 | .175$^5$ | .2411 | 13 49 | .5703 | | | | |
| 0.50 | .4568 | .112 | .168$^5$ | .2315 | 13 16 | .05266 | 0.168$^5$ | 0.000 | 0.169 | —0.000$^5$ |
| 0.55 | .5025 | .208 | .161$^5$ | .2219 | 12 43 | .04846 | | | | |
| 0.60 | .5482 | .313 | .154$^5$ | .2123 | 12 10 | .04442 | 0.154$^5$ | 0.000 | 0.154$^5$ | 0.000 |
| 0.65 | .5939 | .428 | .147$^5$ | .2027 | 11 37 | .04055 | | | | |
| 0.70 | .6396 | .558 | .140 | .1924 | 11 1 | .03652 | 0.139$^5$ | 0.000$^5$ | 0.140 | 0.000 |
| 0.75 | .6853 | .705 | .132$^5$ | .1821 | 10 26 | .03279 | | | | |
| 0.80 | .7310 | .876 | .124 | .1704 | 9 46 | .02878 | 0.123 | 0.001 | 0.123$^5$ | 0.000$^5$ |
| 0.85 | .7766 | 11.077 | .114 | .1566 | 8 58,5 | .02434 | | | | |
| 0.90 | .8223 | .326 | .102 | .1402 | 8 2 | .01953 | 0.102 | 0.000 | 0.102 | 0.000 |
| 0.95 | .8680 | .648 | .087 | .1195 | 6 51 | .01423 | 0.088 | —0.001 | 0.088 | —0.001 |
| 0.98 | .8954 | .901 | .074 | | | | 0.076 | —0.002 | 0.075 | —0.001 |
| 0.99 | .9046 | 12.001 | .068 | | | | 0.071 | —0.003 | 0.068 | 0.000 |
| 1.00 | .9137 | .11 | .058 | | | | 0.061 | —0.003 | 0.049 | 0.009 |

| Parkhurst | | | | | | | Theor. light-curve | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | U-hyp. | | D-hyp. | |
| $a$ | $1-l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | $\tau_c$ | O—C | $\tau_c$ | O—C |
| 0.00 | 0.0000 | $9^m.45$ | $0^d.256$ | | | | $0^d.251$ | $0^d.005$ | $0^d.306$ | $-0^d.050$ |
| 0.10 | .0852 | .574 | .226 | | | | 0.222 | 0.004 | 0.240 | —0.014 |
| 0.20 | .1704 | .710 | .207 | | | | 0.205 | 0.002 | 0.207 | 0.000 |
| 0.25 | .2130 | .783 | .198$^5$ | 0.2727 | 15 37′,5 | 0.07254 | | | | |
| 0.30 | .2556 | .861 | .190$^5$ | .2618 | 15 0 | .06699 | 0.189$^5$ | 0.001 | 0.190 | 0.000$^5$ |
| 0.35 | .2982 | .943 | .182$^5$ | .2508 | 14 22 | .06157 | | | | |
| 0.40 | .3408 | 10.030 | .175 | .2404 | 13 46,5 | .05670 | 0.175 | 0.000 | 0.175 | 0.000 |
| 0.45 | .3834 | .123 | .167$^5$ | .2301 | 13 11 | .05201 | | | | |
| 0.50 | .4260 | .222 | .160$^5$ | .2205 | 12 38 | .04783 | 0.161 | —0.000$^5$ | 0.161 | —0.000$^5$ |
| 0.55 | .4686 | .330 | .153$^5$ | .2109 | 12 5 | .04382 | | | | |
| 0.60 | .5113 | .447 | .146$^5$ | .2013 | 11 32 | .03998 | 0.146$^5$ | 0.000 | 0.147 | —0.000$^5$ |
| 0.65 | .5539 | .574 | .139$^5$ | .1917 | 10 59 | .03630 | | | | |
| 0.70 | .5965 | .713 | .132 | .1814 | 10 23$^5$ | .03254 | 0.131$^5$ | 0.000$^5$ | 0.132 | 0.000 |
| 0.75 | .6391 | .868 | .124 | .1704 | 9 46 | .02878 | | | | |
| 0.80 | .6817 | 11.043 | .115$^5$ | .1587 | 9 5,5 | .02497 | 0.115 | 0.000$^5$ | 0.115$^5$ | 0.000 |
| 0.85 | .7243 | .243 | .106 | .1456 | 8 21 | .02109 | | | | |
| 0.90 | .7669 | .477 | .095 | .1305 | 7 29 | .01696 | 0.095 | 0.000 | 0.095 | 0.000 |
| 0.95 | .8095 | .758 | .082 | .1127 | 6 27 | .01262 | 0.082 | 0.000 | 0.082 | 0.000 |
| 0.98 | .8350 | .959 | .071 | | | | 0.071 | 0.000 | 0.071 | 0.000 |
| 0.99 | .8436 | 12.033 | .066 | | | | 0.066 | 0.000 | 0.064 | 0.002 |
| 1.00 | .8521 | .11 | .058 | | | | 0.057 | 0.001 | 0.048 | 0.010 |

# V 23 = SW CYGNI.

$$P = 4^d.5728\ ; \qquad \frac{2\pi}{P} = 1.374$$

| | A. (Pickering) | | B. (Parkhurst) | |
|---|---|---|---|---|
| | $L_1 = 0.086$ | $L_2 = 0.914$ | $L_1 = 0.148$ | $L_2 = 0.852$ |
| | $\overline{A_1} = 0.06450$ $\quad \overline{A_2} = 0.04055$ | $\overline{A_3} = 0.02172$ | $\overline{A_1} = 0.05961$ $\quad \overline{A_2} = 0.03628$ | $\overline{A_3} = 0.01891$ |
| | $\overline{A_1} - \overline{A_2} = 0.02395$ | $\overline{A_3} - \overline{A_2} = -0.01883.$ | $\overline{A_1} - \overline{A_2} = 0.02333$ | $\overline{A_1} - \overline{A_2} = -0.01737$ |
| | *a* | *b* | *a* | *b* |
| Approximation | Initial value 0.30 final value 0.23 | Initial value 0.40 final value $0.42^7$ | Initial value 0.30 final value $0.34^1$ | Initial value 0.50 final value $0.50^7$ |
| of $k$: | " " 0.25 " " $0.24^8$ | " " 0.42 " " $0.42^0$ | " " 0.33 " " $0.33^1$ | " " $0.50^5$ " " $0.50^6$ |
| Adopted: | $k = 0.25.$ | $k = 0.42.$ | $k = 0.33$ | $k = 0.50^5$ |
| Eq. (4): | $0.93550 \sin^2 i + 1.0826\, r_1^2 = 1$ | $0.93550 \sin^2 i + 1.1072\, r_1^2 = 1$ | $0.94039 \sin^2 i + 1.1018\, r_1^2 = 1$ | $0.94039 \sin^2 i + 1.1881\, r_1^2 = 1$ |
| | $0.97828 \sin^2 i + 0.6962\, r_1^2 = 1$ | $0.97828 \sin^2 i + 0.5577\, r_1^2 = 1$ | $0.98109 \sin^2 i + 0.6050\, r_1^2 = 1$ | $0.98109 \sin^2 i + 0.4763\, r_1^2 = 1$ |
| whence: | $\sin^2 i = 0.9475$ $\quad i = 76°45'$ | $\sin^2 i = 0.9787$ $\quad i = 81°37'$ | $\sin^2 i = 0.9702$ $\quad i = 80°3'$ | $\sin^2 i = 0.9889$ $\quad i = 86°6'$ |
| | $r_1^2 = 0.1048$ $\quad r_1 = 0.324$ | $r_1^2 = 0.0762$ $\quad r_1 = 0.276$ | $r_1^2 = 0.0795$ $\quad r_1 = 0.282$ | $r_1^2 = 0.0627$ $\quad r_1 = 0.250$ |
| | $r_2 = kr_1 = 0.081$ | $r_2 = kr_1 = 0.116$ | $r_2 = kr_1 = 0.093$ | $r_2 = kr_1 = 0.127$ |
| Eq. (5*a*): | Corresponding depth of | Corresponding depth of | Corresponding depth of | Corresponding depth of |
| For D-hyp. (6*a*) | sec. min. $0^m.01$ | sec. min. $0^m.02$ | sec. min. $0^m.02$ | sec. min. $0^m.06$ |
| | $\frac{L_2}{L_1} = 10.6$; therefore $\gamma = 171$ | $\frac{L_2}{L_1} = 10.6$; mean $\gamma = 60$ | $\frac{L_2}{L_1} = 5.8$; therefore $\gamma = 53$ | $\frac{L_2}{L_1} = 5.8$; mean $\gamma = 22.5$ |
| Least apparent distance of centers: | $\cos i = 0.229$ | $\cos i = 0.146$ | $\cos i = 0.173$ | $\cos i = 0.070$ |
| Densities: | $\varrho_1 = 0.0095$; $\varrho_2 = 0.612.$ | $\varrho_1 = 0.015$; $\varrho_2 = 0.206.$ | $\varrho_1 = 0.014$ $\quad \varrho_2 = 0.399$ | $\varrho_1 = 0.020$ $\quad \varrho_2 = 0.158$ |

SUMMARY.

| Depth | | Semi-duration | | | | | | | | | Density | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| prim. | sec. | eclipse | totality | $a_0$ | $L_2$ | $r_1$ | $r_2$ | $k$ | $i$ | $\gamma$ | $\varrho_1$ | $\varrho_2$ | Hypotheses. |
| $2^m.66$ | $0^m.01$ | $0^d.254$ | $0^d.061$ | 1.000 | 0.914 | 0.324 | 0.081 | 0.25 | 76°45′ | 171 | $0.009^5$ | 0.614 | U } Pickering. |
| " | 0.02 | 0.262 | 0.049 | " | " | 0.276 | 0.116 | 0.42 | 81 37 | 60 | 0.015 | 0.206 | D } Pickering. |
| " | 0.02 | 0.251 | 0.057 | " | 0.852 | 0.282 | 0.093 | 0.33 | 80 3 | 53 | 0.014 | 0.399 | U } Parkhurst. |
| " | 0.06 | 0.306 | 0.048 | " | " | 0.250 | 0.127 | $0.50^5$ | 86 6 | 22.5 | $0.020^5$ | 0.158 | D } Parkhurst. |

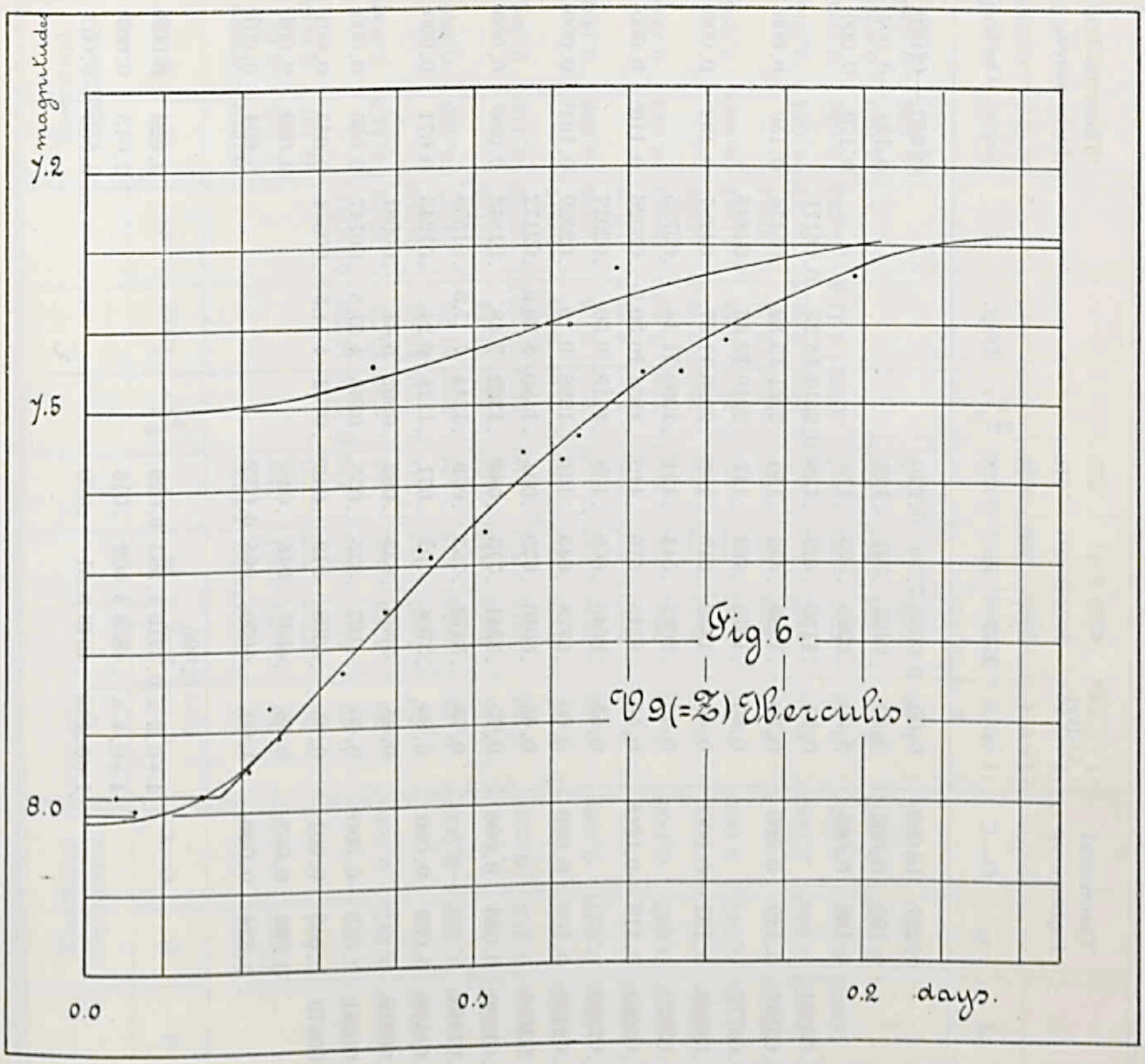
## § 14. V 9 = Z HERCULIS.

| Comparison-stars (1900) | | HP | H′ | Normals | | Light-curve Fig. 6. |
|---|---|---|---|---|---|---|
| α | δ | | | | | |
| $17^h54^m15^s$ | $+14°51.3$ | $7^m.10$ | $7^m.20$ | $-0^d.165$ $7^m.42$ | $+0^d.154$ $7^m.46$ | 0.000 $7^m.98$ ($8^m.01$) |
| 53 48 | 14 31.2 | 7.27 | 7.43 | 0.144 7.46 | 0.123 7.57 | 0.034 7.98 (8.00) |
| 53 32 | 15 25.1 | 8.04 | 7.87 | 0.127 7.54 | 0.086 7.68 | 0.040 $7.95^5$ (7.95) |
| 53 50 | 14 37.4 | 8.50 | 8.50 | 0.113 7.56 | 0.047 7.87 | 0.060 7.85 |
| | | | | 0.103 7.66 | 0.013 $7.99^5$ | 0.080 $7.74^5$ |
| | | | | 0.089 7.69 | (0.198 7.34) | 0.100 7.65 |
| | | | | 0.077 $7.75^5$ | | 0.120 7.56 |
| | | | | 0.066 7.83 | | 0.140 $7.48^5$ |
| | | | | 0.050 7.91 | | 0.160 7.42 |
| | | | | 0.041 7.95 | | 0.180 7.37 |
| | | | | 0.030 7.98 | | 0.200 $7.32^5$ |
| | | | | 0.008 7.98 | | 0.220 7.30 |
| | | | | | | 0.230 7.29 |

In this case it is possible that the eclipse at primary minimum is annular. In the U-hyp. this minimum will show a constant brightness ; whereas in the D-hyp. there is no standstill, the light-curve being indistinguishable from that of a partial eclipse. In tracing the light-curve both possibilities have been admitted near the minimum, the observations being not contradictory to one of these possibilities.

Maximum $7^m.29$. Primary minimum $7^m.98$ ($8^m.01$). Secondary minimum $7^m.50$
Depth $0^m.69$ ($0^m.72$). Depth $0^m.21$

Semi-duration of eclipse $t_1 = 0.225$ ; of constant minimum $t_2 = 0^d.035$. Upper limit of $k$ about $\frac{t_1-t_2}{t_1+t_2} = 0^m.73$. Whether the principal eclipse is total or annular can only be determined when the final value of $k$ has been found.


7 magnitudes
7.2
7.5
8.0
0.0
0.1
0.2 days.
Fig. 6.
V9(=Z) Herculis.

# TABLE 15.

| Pickering. U-hyp. | | | | | | | Theoretical Light-curve | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 1—l | m | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | A | $\tau_c$ | O—C |
| 0.00 | 0.0000 | 7$^m$29 | 0$^d$230 | | | | 0$^d$227 | 0$^d$003 |
| 0.10 | .0470 | .342 | .192 | | | | 0.190 | 0.002 |
| 0.20 | .0941 | .397 | .169 | | | | 0.168 | 0.001 |
| 0.25 | .1176 | .426 | .158$^5$ | 0.2495 | 14°17',5 | 0.06094 | | |
| 0.30 | .1411 | .455 | .149 | .2345 | 13 26 | .05397 | 0.149 | 0.000 |
| 0.35 | .1646 | .485 | .140 | .2204 | 12 37,5 | .04777 | | |
| 0.40 | .1881 | .516 | .131$^5$ | .2070 | 11 51,5 | .04223 | 0.132 | —0.000$^5$ |
| 0.45 | .2116 | .548 | .123$^5$ | .1944 | 11 8 | .03729 | | |
| 0.50 | .2352 | .581 | .115$^5$ | .1818 | 10 25 | .03269 | 0.116 | —0.000$^5$ |
| 0.55 | .2587 | .615 | .107$^5$ | .1692 | 9 42 | .02839 | | |
| 0.60 | .2822 | .650 | .100 | .1574 | 9 1 | .02456 | 0.100 | 0.000 |
| 0.65 | .3057 | .686 | .092$^5$ | .1456 | 8 20,5 | .02105 | | |
| 0.70 | .3292 | .724 | .085 | .1338 | 7 40 | .01780 | 0.085 | 0.000 |
| 0.75 | .3527 | .762 | .077 | .1212 | 6 57 | .01464 | | |
| 0.80 | .3762 | .802 | .069 | .1086 | 6 13,5 | .01176 | 0.069 | 0.000 |
| 0.85 | .3998 | .844 | .061 | .0960 | 5 30 | .00919 | | |
| 0.90 | .4233 | .888 | .052$^5$ | .0826 | 4 44 | .00681 | 0.053 | —0.000$^5$ |
| 0.95 | .4468 | .933 | .044 | .0693 | 3 58 | .00479 | 0.043 | 0.001 |
| 0.98 | .4609 | .961 | .039 | | | | 0.036 | 0.003 |
| 1.00 | .4703 | .98 | .034 | | | | 0.028 | 0.006 |

| D-Hyp. | | | | | | | Theoretical Light-curve. | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 1—l | m | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | A | $\tau_c$ | O—C |
| 0.00 | 0.0000 | 7$^m$29 | 0$^d$230 | | | | 0$^d$250 | —0$^d$020 |
| 0.10 | .0463 | .341 | .192 | | | | 0.195 | —0.003 |
| 0.20 | .0926 | .395 | .170 | | | | 0.170 | 0.000 |
| 0.25 | .1156 | .423 | .159$^5$ | 0.2510 | 14°23' | 0.06171 | | |
| 0.30 | .1389 | .452 | .150 | .2361 | 13 32 | .05476 | 0.150 | 0.000 |
| 0.35 | .1620 | .482 | .141 | .2219 | 12 43 | .04846 | | |
| 0.40 | .1852 | .513 | .132$^5$ | .2086 | 11 57 | .04287 | 0.132$^5$ | 0.000 |
| 0.45 | .2084 | .544 | .124$^5$ | .1960 | 11 14 | .03795 | | |
| 0.50 | .2315 | .576 | .116$^5$ | .1834 | 10 30,5 | .03326 | 0.116$^5$ | 0.000 |
| 0.55 | .2546 | .609 | .109 | .1716 | 9 50 | .02917 | | |
| 0.60 | .2778 | .643 | .101$^5$ | .1598 | 9 9 | .02529 | 0.101$^5$ | 0.000 |
| 0.65 | .3010 | .679 | .094 | .1480 | 8 28,5 | .02172 | | |
| 0.70 | .3241 | .715 | .086$^5$ | .1362 | 7 48 | .01842 | 0.086$^5$ | 0.000 |
| 0.75 | .3472 | .753 | .079 | .1244 | 7 7,5 | .01538 | | |
| 0.80 | .3704 | .792 | .071 | .1118 | 6 24 | .01243 | 0.071 | 0.000 |
| 0.85 | .3936 | .833 | .063 | .0992 | 5 41 | .00981 | | |
| 0.90 | .4167 | .875 | .055 | .0866 | 4 57,5 | .00747 | 0.055 | 0.000 |
| 0.95 | .4398 | .919 | .046$^5$ | .0732 | 4 11,5 | .00534 | 0.045 | 0.001$^5$ |
| 0.98 | .4537 | .946 | .040$^5$ | | | | 0.039 | 0.001$^5$ |
| 1.00 | .4630 | .965 | 0.035$^5$ | | | | 0.034 | 0.001$^5$ |
| x = 0.096 | | | | | | | | |
| 1+0.2 x | 0.4719 | 7.983 | 0.029 | | | | 0.026 | 0.003 |
| 1+0.4 x | .4808 | 8.002 | .018 | | | | 0.015 | 0.003 |
| 1+0.49x | .4848 | 8.01 | .000 | | | | 0.003 | —0.003 |

## TABLE 16.

| Parkhurst. U-hyp. | | | | | | | Theoretical Light-curve | | D-Hyp. | | | | | | | Theoretical Light-curve | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 1—$l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | $\tau_c$ | O—C | $\alpha$ | 1—$l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | $\tau_c$ | O—C |
| 0.00 | 0.0000 | $7^{m}29$ | $0^{d}230$ | | | | $0^{d}226$ | $0^{d}004$ | 0.00 | 0.0000 | $7^{m}29$ | $0^{d}230$ | | | | $0^{d}249$ | $-0^{d}019$ |
| 0.10 | .0391 | .345 | .191 | | | | 0.188 | 0.003 | 0.10 | .0463 | .341 | .192 | | | | 0.192 | —0.001 |
| 0.20 | .0782 | .403 | $.166^5$ | | | | 0.166 | $0.000^5$ | 0.20 | .0926 | .395 | .170 | | | | 0.167 | $0.000^5$ |
| 0.25 | .0977 | .433 | .156 | 0.2455 | 14° 4′ | 0.05907 | | | 0.25 | .1158 | .423 | $.159^5$ | 0.2510 | 14°23′ | 0.06171 | | |
| 0.30 | .1173 | .464 | .147 | .2314 | 13 15′,5 | .05260 | 0.147 | 0.000 | 0.30 | .1389 | .452 | .150 | .2361 | 13 32 | .05476 | $0.147^5$ | 0.000 |
| 0.35 | .1368 | .495 | .138 | .2172 | 12 27 | .04648 | | | 0.35 | .1620 | .482 | .141 | .2219 | 12 43 | .04846 | | |
| 0.40 | .1564 | .527 | .129 | .2030 | 11 38 | .04066 | $0.129^5$ | $-0.000^5$ | 0.40 | .1852 | .513 | $.132^5$ | .2086 | 11 57 | .04287 | 0.130 | 0.000 |
| 0.45 | .1759 | .559 | $.120^5$ | .1896 | 10 52 | .03554 | | | 0.45 | .2084 | .544 | $.124^5$ | .1960 | 11 14 | .03795 | | |
| 0.50 | .1954 | .593 | $.112^5$ | .1772 | 10 9 | .03106 | 0.113 | $-0.000^5$ | 0.50 | .2315 | .576 | $.116^5$ | .1834 | 10 30,5 | .03326 | 0.114 | 0.000 |
| 0.55 | .2150 | .627 | .105 | .1653 | 9 28 | .02705 | | | 0.55 | .2546 | .609 | .109 | .1716 | 9 50 | .02917 | | |
| 0.60 | .2345 | .662 | $.097^5$ | .1535 | 8 47′,5 | .02336 | $0.097^5$ | 0.000 | 0.60 | .2778 | .643 | $.101^5$ | .1598 | 9 9 | .02529 | 0.099 | 0.000 |
| 0.65 | .2541 | .698 | .090 | .1417 | 8 7 | .01993 | | | 0.65 | .3010 | .679 | .094 | .1480 | 8 28,5 | .02172 | | |
| 0.70 | .2736 | .735 | $.082^5$ | .1298 | 7 26′,5 | .01677 | 0.082 | $0.000^5$ | 0.70 | .3241 | .715 | $.086^5$ | .1362 | 7 48 | .01842 | 0.084 | $0.000^5$ |
| 0.75 | .2932 | .773 | .075 | .1180 | 6 46 | .01388 | | | 0.75 | .3472 | .753 | .079 | .1244 | 7 7,5 | .01538 | | |
| 0.80 | .3127 | .812 | $.067^5$ | .1062 | 6 5 | .01123 | 0.067 | $0.000^5$ | 0.80 | .3704 | .792 | .071 | .1118 | 6 24 | .01243 | $0.069^5$ | 0.000 |
| 0.85 | .3323 | .852 | $.059^5$ | .0936 | 5 22 | .00875 | | | 0.85 | .3936 | .833 | .063 | .0992 | 5 41 | .00981 | | |
| 0.90 | .3518 | .893 | $.051^5$ | .0811 | 4 39 | .00657 | $0.051^5$ | 0.000 | 0.90 | .4167 | .875 | .055 | .0866 | 4 57,5 | .00747 | 0.054 | 0.000 |
| 0.95 | .3714 | .936 | $.043^5$ | .0685 | 3 55,5 | .00469 | 0.043 | $0.000^5$ | 0.95 | .4398 | .919 | $.046^5$ | .0732 | 4 11,5 | .00534 | 0.046 | 0.000 |
| 0.98 | .3831 | .963 | .038 | | | | $0.036^5$ | $0.001^5$ | 0.98 | .4537 | .946 | $.040^5$ | | | | 0.040 | $0.000^5$ |
| 1.00 | .3909 | .98 | .034 | | | | 0.030 | 0.004 | 1.00 | .4630 | .965 | $.035^5$ | | | | 0.036 | $-0.000^5$ |
| | | | | | | | | | x = | 0.70 | | | | | | | |
| | | | | | | | | | 1+0.2 x | 0.3894 | 7.977 | 0.032 | | | | $0.031^5$ | $0.000^5$ |
| | | | | | | | | | 1+0.4 x | .3948 | .989 | $.026^5$ | | | | 0.027 | $-0.000^5$ |
| | | | | | | | | | 1+0.6 x | .4001 | 8.001 | .019 | | | | 0.021 | —0.002 |
| | | | | | | | | | 1+0.74x | .4039 | 8.01 | .000 | | | | 0.009 | —0.009 |

In the supposition A,b) a total eclipse at primary minimum is still much less probable than in the U-hyp.; $k$ coming out about 0.90, so that $\alpha_0$ remains far below 1.00. We, therefore, assume immediately an angular eclipse and, in accordance herewith, we alter the light-curve near minimum, because now there cannot be a constant minimum light. As we have remarked already the observations are not at variance with this possibility. The magnitude at primary minimum is now $8^{m}.01$ and the range of variation $0^{m}.72$.

In this case the loss of light at internal contact is to be chosen as unit of light. Supposing the semi-duration of the annular eclipse to be at least as long as that given by the U-hyp., we find for the magnitude at the moment of internal contact $7^{m}.965$ and for the loss of light at that moment 0.4630.

The same value would result from the following consideration: In order to determine the ratio $\nu$ of the loss of light at internal contact to that at mid-eclipse, we divided a circle with radius 1, into $10 \times 32$ parts from which, according to the adopted law of darkening, equal amounts of light would be received[1]). Upon this circle we superpose a circle with radius $k$, whose center passes the center of the first circle at the distance $\cos i$; now the divisions covered by the second circle are counted, fractions of divisions being estimated. Probably a good approximation for the values of $k$ and $\cos i$ may be furnished by the U-hyp., since the changes in $k$ and $i$ caused by the D-hyp. have, as a rule, opposite effects on the values of $\nu$. Repeating the procedure for the case of internal contact, $\nu$ comes out to be about 0.953, from which $\alpha_0'' = 1.047$. Since the loss of light at minimum is 0.4848, the loss at the moment of internal contact is therefore about 0.4620. We have adopted in Table 15 the value 0.4630.

Computing, in the same manner as above, but now with the derived values of $r_1$, $k$ and $i$, the fraction $\nu$, we get $\nu = 0.949$ and therefore the loss of light at the moment of internal contact 0.4600.

If the light-curve of the secondary minimum is well-determined, the semi-duration of total eclipse at that minimum and, therefore, the moment of internal contact at primary minimum, the magnitude and the loss of light at that moment can immediately be found.

In the supposition B,b) for the magnitude at the moment of internal contact we also take $7^{m}.965$; hence the loss of light at that moment amounts to 0.3840 and $\alpha_0'' = 1.052$.

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1) See Ap.J. **36**, 241 (1912).

# V 9 = Z HERCULIS.

$P' = 3.9928 \qquad \frac{2\pi}{P} = 1.574.$

| | A. (Pickering) a | A. (Pickering) b | B. (Parkhurst) a | B. (Parkhurst) b |
|---|---|---|---|---|
| | $\overline{A}_1 = 0.04582$ ; $\overline{A}_2 = 0.02129$ ; $\overline{A}_3 = 0.00814$ | $\overline{A}_1 = 0.04650$ ; $\overline{A}_2 = 0.02200$ ; $\overline{A}_3 = 0.00876$ | $\overline{A}_1 = 0.04424$ ; $\overline{A}_2 = 0.02020$; $\overline{A}_3 = 0.00781$ | $\overline{A}_1 = 0.04485$ ; $\overline{A}_2 = 0.02094$ ; $\overline{A}_3 = 0.00846$ |
| | $\overline{A}_1-\overline{A}_2=0.02453$; $\overline{A}_3-\overline{A}_2=-0.01315$ | $\overline{A}_1-\overline{A}_2=0.02450$; $\overline{A}_3-\overline{A}_2=-0.01324$ (see p. 70) | $\overline{A}_1-\overline{A}_2=0.02404$; $\overline{A}_3-\overline{A}_2=-0.01239$ | $\overline{A}_1-\overline{A}_2=0.02391$; $\overline{A}_3-\overline{A}_2=-0.01248$ (see p. 70) |
| Approximation of $k$: | Initial value 0.70 final value 0.736<br>„ „ 0.725 „ „ 0.726 | Initial value 0.70 final value 0.683 (Table III)<br>„ „ 0.68 „ „ 0.680 | Initial value 0.75 final value 0.766<br>„ „ 0.76 „ „ 0.761 | Initial value 0.75 final value 0.742 (Table III)<br>„ „ 0.74 „ „ 0.739 |
| Adopted: | $k = 0.725$ | $k = 0.68$ | $k = 0.76$ | $k = 0.74$ |
| Eq. (4): | $0.95418 \sin^2 i + 1.1366\, r_1^2 = 1$<br>$0.99186 \sin^2 i + 0.2255\, r_1^2 = 1$ | $0.95350 \sin^2 i + 0.9265\, r_1^2 = 1$<br>$0.99124 \sin^2 i + 0.2146\, r_1^2 = 1$ | $0.95576 \sin^2 i + 1.1347\, r_1^2 = 1$<br>$0.99219 \sin^2 i + 0.1980\, r_1^2 = 1$ | $0.95515 \sin^2 i + 0.9355\, r_1^2 = 1$<br>$0.99154 \sin^2 i + 0.1790\, r_1^2 = 1$ |
| whence: | $\sin^2 i = 0.9988$ $i = 88°0'$<br>$r_1^2 = 0.0413$ $r_1 = 0.203$<br>$r_2 = kr_1 = 0.148$ | $\sin^2 i = 0.9974$ $i = 87°4'$<br>$r_1^2 = 0.0529$ $r_1 = 0.230$<br>$r_2 = kr_1 = 0.156$ | $\sin^2 i = 1.0001$ $i = 90°$<br>$r_1^2 = 0.0391$ $r_1 = 0.198$<br>$r_2 = kr_1 = 0.150$ | $\sin^2 i = 0.9999$ $i = 89°20'$<br>$r_1^2 = 0.0481$ $r_1 = 0.219$<br>$r_2 = kr_1 = 0.162$ |
| Eq. (5a) (Case E1): | $\alpha_0 = 0.4703 + \frac{0.1759}{0.527} = 0.805$ | | $\alpha_0 = 0.3909 + \frac{0.1400}{0.579} = 0.633$ | |
| Eq. (5b) (Case Es) | $\alpha_0 = 0.1759 + \frac{0.4703}{0.527} = 1.067$ | $\alpha_0' = 0.1759 + \frac{0.4848}{0.580} = 1.012,$ | $\alpha_0 = 0.1400 + \frac{0.3909}{0.579} = 0.815$ | $\alpha_0' = 0.1400 + \frac{0.4038}{0.683} = 0.731,$ |
| For the D-hyp. eq. (6b). | As in both cases $\alpha_0$ must be unity, the latter case (annular eclipse at primary minimum) seems to be most probable. | in satisfactory agreement with the theoretical value 1.00. | The eclipse at primary minimum must therefore be assumed as annular. | a very discordant value. |
| $L_1$: | According to the depth of primary minimum $\frac{0.4713}{0.527} = 0.892$; according to the depth of secondary minimum 0.824. Adopted $L_1=0.88$ and $L_2=0.12$. Corresponding depth of secondary minimum $0^m.14$. | According to the depth of primary minimum $\frac{0.4848}{0.580} = 0.836$; according to the depth of secondary minimum 0.824. Adopted $L_1=0.83$ and $L_2=0.17$. Corresponding depth of secondary minimum $0^m.20$. | According to the depth of primary minimum $\frac{0.3909}{0.579} = 0.675$; according to the depth of secondary minimum 0.860. Adopted $L_1=0.70$ and $L_2=0.30$. Corresponding depth of secondary minimum $0^m.50$. | According to the depth of primary minimum $\frac{0.4038}{0.683} = 0.591$; according to the depth of secondary minimum 0.860. Adopted $L_1=0.65$ and $L_2=0.35$. Corresponding depth of secondary minimum $0^m.60$. |
| | $\frac{L_1}{L_2} = 7.3$; therefore $\gamma = 0.26$. | $\frac{L_1}{L_2} = 4.9$; mean $\gamma = 0.43$. | $\frac{L_1}{L_2} = 2.4$; therefore $\gamma = 0.71$. | $\frac{L_1}{L_2} = 1.9$; mean $\gamma = 0.90$. |
| Least apparent distance of centers: | $\cos i = 0.035.$ | $\cos i = 0.051.$ | $\cos i = 0.000.$ | $\cos i = 0.012.$ |
| Densities: | $\varrho_1 = 0.050$; $\varrho_2 = 0.131.$ | $\varrho_1 = 0.035$; $\varrho_2 = 0.110.$ | $\varrho_1 = 0.054$; $\varrho_2 = 0.124.$ | $\varrho_1 = 0.040$; $\varrho_2 = 0.099.$ |

## V 9 = Z HERCULIS.

SUMMARY.

| Depth | | Semi-duration | | | | | | | | | Density | | Hypotheses |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| prim. | sec. | eclipse | totality | $a_0$ ($a_0''$) | $L_1$ | $r_1$ | $r_2$ | $k$ | $i$ | $\gamma$ | $\varrho_1$ | $\varrho_2$ | |
| $0^m.69$ | $0^m.14$ | $0^d.227$ | $0^d.028$ | 1.00 | 0.88 | 0.203 | 0.148 | $0.72^5$ | 88° 0′ | 0.26 | 0.050 | 0.131 | U } PICKERING. |
| 0.72 | 0.20 | 0.250 | 0.034 | 1.047 | 0.83 | 0.230 | 0.156 | 0.68 | 87° 4′ | 0.43 | 0.035 | 0.110 | D } PICKERING. |
| 0.69 | 0.50 | 0.226 | 0.030 | 1.00 | 0.70 | 0.198 | 0.150 | 0.76 | 90° | 0.71 | 0.054 | 0.124 | U } PARKHURST. |
| 0.72 | 0.60 | 0.249 | 0.036 | 1.052 | 0.65 | 0.219 | 0.162 | 0.74 | 89°20′ | 0.90 | 0.040 | 0.099 | D } PARKHURST. |

*Remark*: It is to be regretted that the light-curve of *V* 9 = *Z Herculis* is very inaccurately known as yet. The reasons are to be found not only in the small range of light-variation, but also in the fact that the period is almost exactly 4 days.

If its light-curve were known with considerable accuracy, this system might give most valuable information about the controversy between the photometric systems of PICKERING and PARKHURST. In particular the depth of the secondary minimum might be decisive (see summary). In general, systems with total (annular) eclipse, a shallow primary minimum and a well-determinable secondary minimum, might be used with success.

The results just found for *V* 9 = *Z Herculis* point strongly to the exactness of the scale of PICKERING.

A system with an annular eclipse at primary minimum — as *V* 9 = *Z Herculis* — may also give some information about the question whether, and to what extent, a darkening towards the limb takes place, if the magnitudes during the annular phases can be observed accurately.

## § 15. V 12 = RT PERSEI.

| Comparison-stars (1900) | | HP | H′ | Normals | | | | Light-curve Fig. 7. | |
|---|---|---|---|---|---|---|---|---|---|
| α | δ | | | | | | | | |
| $3^h15^m52^s$ | +46°18′ | $10^m.21$ | $10^m.20$ | $-0^d.057^5$ | $10^m.62$ | $+0^d.061^5$ | $10^m.56^5$ | 0.000 | 11.90 |
| 16 14 | 1 | 10.81 | 10.80 | $0.043^5$ | 10.82 | $0.051^5$ | $10.66^5$ | (0.005) | (11.89) |
| 16 19 | 18.5 | 11.25 | 11.30 | $0.036^5$ | 10.98 | $0.044^5$ | 10.82 | 0.007 | 11.90 |
| 16 28 | 7 | 11.82 | 11.96 | $0.029^5$ | 11.31 | $0.038^5$ | 10.93 | 0.010 | (11.85) |
| | | | | $0.022^5$ | $11.50^5$ | $0.033^5$ | 11.18 | | 11.86 |
| | | | | $0.015^5$ | 11.74 | $0.027^5$ | 11.39 | 0.015 | (11.75) |
| | | | | $0.010^5$ | $11.84^5$ | $0.020^5$ | 11.66 | | 11.74 |
| | | | | 0.006 | 11.91 | $0.015^5$ | 11.70 | 0.020 | (11.61) |
| | | | | $0.001^5$ | $11.90^5$ | $0.009^5$ | $11.82^5$ | | 11.60 |
| | | | | | | $0.004^5$ | 11.88 | 0.025 | 11.44 |
| | | | | | | | | 0.030 | 11.26 |
| | | | | | | | | 0.035 | $11.07^5$ |
| | | | | | | | | 0.040 | 10.92 |
| | | | | | | | | 0.045 | 10.80 |
| | | | | | | | | 0.050 | 10.71 |
| | | | | | | | | 0.055 | 10.64 |
| | | | | | | | | 0.060 | 10.58 |
| | | | | | | | | 0.070 | 10.49 |
| | | | | | | | | 0.080 | 10.44 |
| | | | | | | | | 0.085 | 10.42 |
| | | | | | | | | 1.000 | 10.42 |

Maximum $10^m.42$. Primary minimum $11^m.90$. Secondary minimum $10^m.53$.
Depth $1^m.48$. Depth $0^m.11$.

From the course of the light-curve and the depth of the minima we may expect a nearly total eclipse.

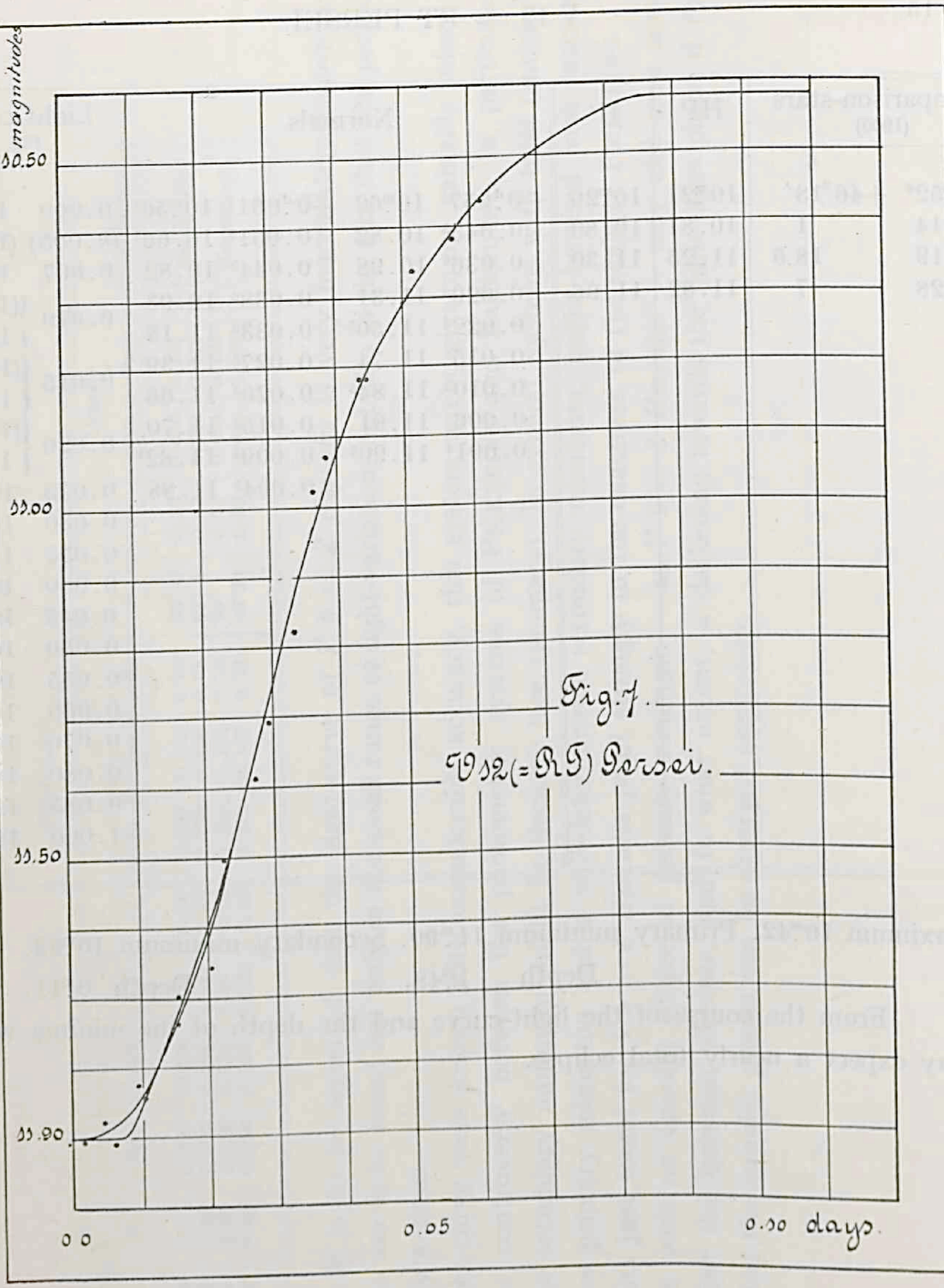

magnitudes
10.50
11.00
11.50
11.90
0.0
0.05
0.10 days
Fig. 7.
V 12 (= RT) Persei.

TABLE 17.

**Pickering**

| $a$ | 1—$l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | Theor. Light-curve U-hyp. $\tau_c$ | O—C | D-hyp. $\tau_c$ | O—C |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | $10^{m}42$ | $0^{d}085$ | | | | $0^{d}075$ | $0^{d}010$ | $0^{d}082$ | $0^{d}003$ |
| 0.10 | .0744 | .504 | .068 | | | | 0.063$^5$ | 0.004$^5$ | 0.065 | 0.003 |
| 0.20 | .1488 | .595 | .058$^5$ | | | | 0.056 | 0.002$^5$ | 0.057 | 0.001$^5$ |
| 0.25 | .1860 | .643 | .054$^{75}$ | 0.4050 | 23°12' | 0.15519 | | | | |
| 0.30 | .2232 | .694 | .051 | .3772 | 21 37 | .13571 | 0.051 | 0.000 | 0.051 | 0.000 |
| 0.35 | .2604 | .747 | .048 | .3551 | 20 20,5 | .12084 | | | | |
| 0.40 | .2976 | .804 | .045 | .3329 | 19 4 | .10671 | 0.045$^5$ | —0.000$^5$ | 0.045$^5$ | —0.000$^5$ |
| 0.45 | .3348 | .863 | .042$^{25}$ | .3125 | 17 54,5 | .09455 | | | | |
| 0.50 | .3720 | .925 | .039$^{75}$ | .2940 | 16 51 | .08402 | 0.040$^5$ | —0.000$^{75}$ | 0.040 | —0.000$^{25}$ |
| 0.55 | .4092 | .992 | .037$^5$ | .2774 | 15 53,5 | .07498 | | | | |
| 0.60 | .4465 | 11.062 | .035$^{25}$ | .2607 | 14 56 | .06641 | 0.035$^5$ | —0.000$^{25}$ | 0.035$^5$ | —0.000$^{25}$ |
| 0.65 | .4837 | .138 | .033 | .2441 | 13 59 | .05839 | | | | |
| 0.70 | .5209 | .219 | .031 | .2293 | 12 8 | .05163 | 0.030$^5$ | 0.000$^5$ | 0.031 | 0.000 |
| 0.75 | .5581 | .308 | .028$^5$ | .2108 | 12 5 | .04382 | | | | |
| 0.80 | .5953 | .402 | .026 | .1923 | 11 1 | .03652 | 0.025$^5$ | 0.000$^5$ | 0.025$^5$ | 0.000$^5$ |
| 0.85 | .6325 | .507 | .023 (.023$^{25}$) | .1701 (.1720) | 9 45 (9 5) | .02868 (.02926) | | | | |
| 0.90 | .6697 | .623 | .019 (.019$^{75}$) | .1405 (.1461) | 8 3 (8 22) | .01961 (.02117) | 0.019$^5$ | —0.000$^5$ | 0.019$^5$ | —0.000$^5$ |
| 0.95 | .7069 | .753 | .014$^5$ (.015) | .1073 (.1107) | 6 9 (6 22,5) | .01148 (.01226) | 0.015$^5$ | —0.001 | 0.015$^5$ | —0.001 |
| 0.98 | .7272 | .838 | .011 (.010$^5$) | | | | 0.012$^5$ | —0.001$^5$ | 0.012$^5$ | —0.001$^5$ |
| 0.99 | .7366 | .869 | .009$^5$ (.008) | | | | 0.011 | —0.001$^5$ | 0.011 | —0.001$^5$ |
| 1.00 | .7441 | 11.90 | .007$^5$ (.000) | | | | 0.008$^5$ | —0.001 | 0.007$^5$ | 0.000 |

**Parkhurst**

| $a$ | 1—$l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | Degr. | $A$ | Theor. Light-curve U-hyp. $\tau_c$ | O—C | D-hyp. $\tau_c$ | O—C |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | $10^{m}42$ | $0^{d}085$ | | | | $0^{d}072$ | $0^{d}013$ | $0^{d}079$ | $0^{d}006$ |
| 0.10 | 0.0655 | .514 | .067 | | | | 0.061 | 0.006 | 0.062 | 0.005 |
| 0.20 | 0.1309 | .615 | .057 | | | | 0.054 | 0.003 | 0.055 | 0.002 |
| 0.25 | 0.1637 | .669 | .053 | 0.3920 | 22°28' | 0.14604 | | | | |
| 0.30 | 0.1964 | .724 | .049$^{25}$ | .4643 | 20 52',5 | .12658 | 0.049 | 0.000$^{25}$ | 0.050 | —0.000$^{75}$ |
| 0.35 | 0.2291 | .782 | .046 | .3403 | 19 30 | .11143 | | | | |
| 0.40 | 0.2619 | .843 | .043 | .3181 | 18 13,5 | .09781 | 0.043$^{75}$ | —0.000$^{75}$ | 0.043$^5$ | —0.000$^5$ |
| 0.45 | 0.2946 | .906 | .040$^5$ | .2996 | 17 10 | .08712 | | | | |
| 0.50 | 0.3274 | .972 | .038$^{25}$ | .2829 | 16 13 | .07799 | 0.039 | —0.000$^{75}$ | 0.038$^{75}$ | —0.000$^5$ |
| 0.55 | 0.3601 | 11.042 | .036 | .2663 | 15 15,5 | .06926 | | | | |
| 0.60 | 0.3928 | .114 | .034 | .2515 | 14 24,5 | .06192 | 0.034 | 0.000 | 0.034 | 0.000 |
| 0.65 | 0.4256 | .192 | .031$^{75}$ | .2348 | 13 27,5 | .05416 | | | | |
| 0.70 | 0.4583 | .273 | .029$^5$ | .2182 | 12 30 | .04685 | 0.029$^5$ | 0.000 | 0.029 | 0.000$^5$ |
| 0.75 | 0.4910 | .360 | .027$^{25}$ | .2016 | 11 33 | .04009 | | | | |
| 0,80 | 0.5238 | .453 | .024$^{75}$ | .1831 | 10 29,5 | .03316 | 0.024$^{25}$ | 0.000$^5$ | 0.024 | 0.000$^{75}$ |
| 0.85 | 0.5565 | .552 | .021$^{75}$ | .1609 | 9 13 | .02565 | | | | |
| 0.90 | 0.5892 | .658 | .018 | .1331 | 7 38 | .01764 | 0.018$^5$ | —0.000$^5$ | 0.018 | 0.000 |
| 0.95 | 0.6220 | .774 | .013$^5$ | .0999 | 5 43 | .00992 | 0.014$^5$ | —0.001 | 0.014 | —0.000$^5$ |
| 0.98 | 0.6416 | .848 | .010$^5$ | | | | 0.011$^5$ | —0.001 | 0.010$^{75}$ | —0.000$^{25}$ |
| 0.99 | 0.6482 | .874 | .009 | | | | 0.010 | —0.001 | 0.009 | 0.000 |
| 1.00 | 0.6547 | 11.90 | .007$^5$ | | | | 0.007 | 0.000$^5$ | 0.003$^5$ | 0.004 |

$P = 0^d\!.8494$ ; $\frac{2\pi}{P} = 7.397$.

**A.** (Pickering)

$\overline{A_1} = 0.11617$ ; $\overline{A_2} = 0.05905$ ; $\overline{A_3} = 0.02480$.

$\overline{A_1} - \overline{A_2} = 0.05712$ ; $\overline{A_3} - \overline{A_2} = -0.03498$.

**B.** (Parkhurst)

$\overline{A_1} = 0.10783$ ; $\overline{A_2} = 0.05446$ ; $\overline{A_3} = 0.02228$.

$\overline{A_1} - \overline{A_2} = 0.05337$ ; $\overline{A_3} - \overline{A_2} = -0.03218$

| | A, a | A, b | B, a |
|---|---|---|---|
| | $\alpha_0 = 0.7441 + \frac{0.0964}{k^2}$ (5a) or $\alpha_0 = 0.0964 + \frac{0.7441}{k^2}$ (5b); $\alpha_0 > 0.84$ | $\alpha_0' = 0.7441 + \frac{0.0964}{Q(k,\alpha_0')}$ (6a) or $\alpha_0' = 0.0964 + \frac{0.7441}{Q(k,\alpha_0')}$ (6b); $\alpha_0' > 0.84$ | $\alpha_0' = 0.6547 + \frac{0.0760}{k^2}$ (5a). or $\alpha_0' = 0.0760 + \frac{0.6547}{k^2}$ (5b). $\alpha_0 > 0.73$ |
| Approximation: of $k$: | From (5a) From (5b) From light-curve<br>$\alpha_0$=1.00: $k$=0.614; $k$=0.908; $k$=0.612 | Form (6a) Form (6b) Form light-curve<br>$\alpha_0'$=1.00: $k$=0.56; $k$=0.85; $k$=0.794<br>$\alpha_0'$=0.90: $k$=0.76; $k$=0.92; $k$=0.787<br>$\alpha_0'$=0.89: $k$=0.786; $k$=0.787 | Form (5a) Form (5b) Form light-curve<br>$\alpha_0$=1.00: $k$=0.47; $k$=0.84; $k$=0.612<br>$\alpha_0$=0.80: $k$=0.724; $k$=0.95; $k$=0.657<br>$\alpha_0$=0.84: $k$=0.641; $k$=0.640 |
| Adopted: | $\alpha_0$=1.00; $k$=0.61. | $\alpha_0'$=0.89; $k$=0.79 | $\alpha_0$=0.84; $k$=0.64 |
| | The eclipse at primary minimum is just total. | At primary minimum the small star is partially eclipsed. | At primary minimum the small star is partially eclipsed. |
| Eq. (4): | $0.88383 \sin^2 i + 1.1374\, r_1^2 = 1$<br>$0.97520 \sin^2 i + 0.3224\, r_1^2 = 1$<br>$\sin^2 i = 0.9889$ $i = 83°57'$<br>$r_1^2 = 0.1109$ $r_1 = 0.333$<br>$r_2 = kr_1 = 0.204$ | $0.88383 \sin^2 i + 1.2332\, r_1^2 = 1$<br>$0.97520 \sin^2 i + 0.3749\, r_1^2 = 1$<br>$\sin^2 i = 0.9853$ $i = 83°13'$<br>$r_1^2 = 0{,}1049$ $r_1 = 0.324$<br>$r_2 = kr_1 = 0.255$ | $0.89217 \sin^2 i + 1.2826\, r_1^2 = 1$<br>$0.97772 \sin^2 i + 0.4789\, r_1^2 = 1$<br>$\sin^2 i = 0.9722$ $i = 80°24'$<br>$r_1^2 = 0.1035$ $r_1 = 0.322$<br>$r_2 = kr_1 = 0.206$ |

Hitherto we have supposed that we had to deal with a partial, though nearly total eclipse. If, however, we
compute the theoretical light-curve from the derived elements, we see that its lower part (magn. > $11^m\!.70$) is de-
viating from the observed curve in each of the four suppositions, so as to suggest a total eclipse (semi-duration
of totality ranging from $0^d\!.010$—$0^d\!.012$ in the four suppositions). From the fact that $r_2 + \cos i < r_1$ it follows once
more that the eclipse is sensibly total. Since the observations do not contradict this alternative, we have slightly
modified the lower part of the observed curve in the required sense. Of course the new elements are nearly the same
as before (see p. 77) but the theoretical light-curve now agrees with the observed curve throughout.

The first solutions for the suppositions A,b) and B,a) we have added here, as examples of the processes
mentioned in § 9. In table 17 the results of the first computations have been written between brackets.

# V 12 = RT PERSEI.

$P = 0^d.8494 \qquad \frac{2\pi}{P} = 7.397$

**A.** (Pickering)

$L_1 = 0.256 \qquad L_2 = 0.744$

$\bar{A}_1 = 0.11617\ ; \qquad \bar{A}_2 = 0.05905\ ; \qquad \bar{A}_3 = 0.02407$

$\bar{A}_1 - \bar{A}_2 = 0.5712\ ; \qquad \bar{A}_3 - \bar{A}_2 = -0.03498$

**B.** (Parkhurst)

$L_1 = 0.345 \qquad L_2 = 0.655$

$\bar{A}_1 = 0.10783\ ; \qquad \bar{A}_2 = 0.05446\ ; \qquad \bar{A}_3 = 0.02159$

$\bar{A}_1 - \bar{A}_2 = 0.05337 \qquad \bar{A}_3 - \bar{A}_2 = -0.03287$

| | A. a | A. b | B. a | B. b |
|---|---|---|---|---|
| Approximation of $k$: | Initial value 0.60 final value 0.587 | Initial value 0.80 Finial value 0.758 | Initial value 0.60 final value 0.574 | Initial value 0.77 Final value 0.759 |
| | „ „ 0.59 „ „ 0.590 | „ „ 0.77 „ „ 0.768 | „ „ 0.58 „ „ 0.582 | „ „ 0.76 „ „ 0.762 |
| Adopted: | $k = 0.59.$ | $k = 0.77.$ | $k = 0.58.$ | $k = 0.76.$ |
| Eq. (4): | $0.88383 \sin^2 i + 1.1363\, r_1^2 = 1$<br>$0.79593 \sin^2 i + 0.3439\, r_1^2 = 1$ | $0.88383 \sin^2 i + 1.1295\, r_1^2 = 1$<br>$0.97593 \sin^2 i + 0.2544\, r_1^2 = 1$ | $0.89217 \sin^2 i + 1.1362\, r_1^2 = 1$<br>$0.97841 \sin^2 i + 0.3514\, r_1^2 = 1$ | $0.89217 \sin^2 i + 1.1300\, r_1^2 = 1$<br>$0.97841 \sin^2 i + 0.2674\, r_1^2 = 1$ |
| whence: | $\sin^2 i = 0.9844 \quad i = 82°49'$<br>$r_1^2 = 0.1144 \quad r_1 = 0.338$<br>$r_2 = kr_1 = 0.199$ | $\sin^2 i = 0.9973 \quad i = 87°2'$<br>$r_1^2 = 0.1050 \quad r_1 = 0.324$<br>$r_2 = kr_1 = 0.249$ | $\sin^2 i = 0.9834 \quad i = 82°36'$<br>$r_1^2 = 0.1081 \quad r_1 = 0.329$<br>$r_2 = kr_1 = 0.191$ | $\sin^2 i = 0.9949 \quad i = 85°55'$<br>$r_1^2 = 0.0995 \quad r_1 = 0.315$<br>$r_2 = kr_1 = 0.240$ |
| Eq. (5a): | Corresponding depth of sec. min. $0^m.10$. | Corresponding depth of sec. min. $0^m.21$. | Corresponding depth of sec. min. $0^m.17$. | Corresponding depth of sec. min. $0^m.38$. |
| For D-hyp. eq. (5b) | $\frac{L_2}{L_1} = 2.9$; therefore $\gamma = 8.4$ | $\frac{L_2}{L_1} = 2.9$; mean $\gamma = 4.9$. | $\frac{L_2}{L_1} = 1.9$; therefore $\gamma = 5.6$. | $\frac{L_2}{L_1} = 1.9$; mean $\gamma = 3.3$. |
| Least apparent distance of centers: | $\cos i = 0.125$ | $\cos i = 0.052$ | $\cos i = 0.129$ | $\cos i = 0.071$ |
| Densities: | $\varrho_1 = 0.241;\quad \varrho_2 = 1.182$ | $\varrho_1 = 0.274;\quad \varrho_2 = 0.603$ | $\varrho_1 = 0.261;\quad \varrho_2 = 1.336$ | $\varrho_1 = 0.298;\quad \varrho_2 = 0.674$ |

SUMMARY.

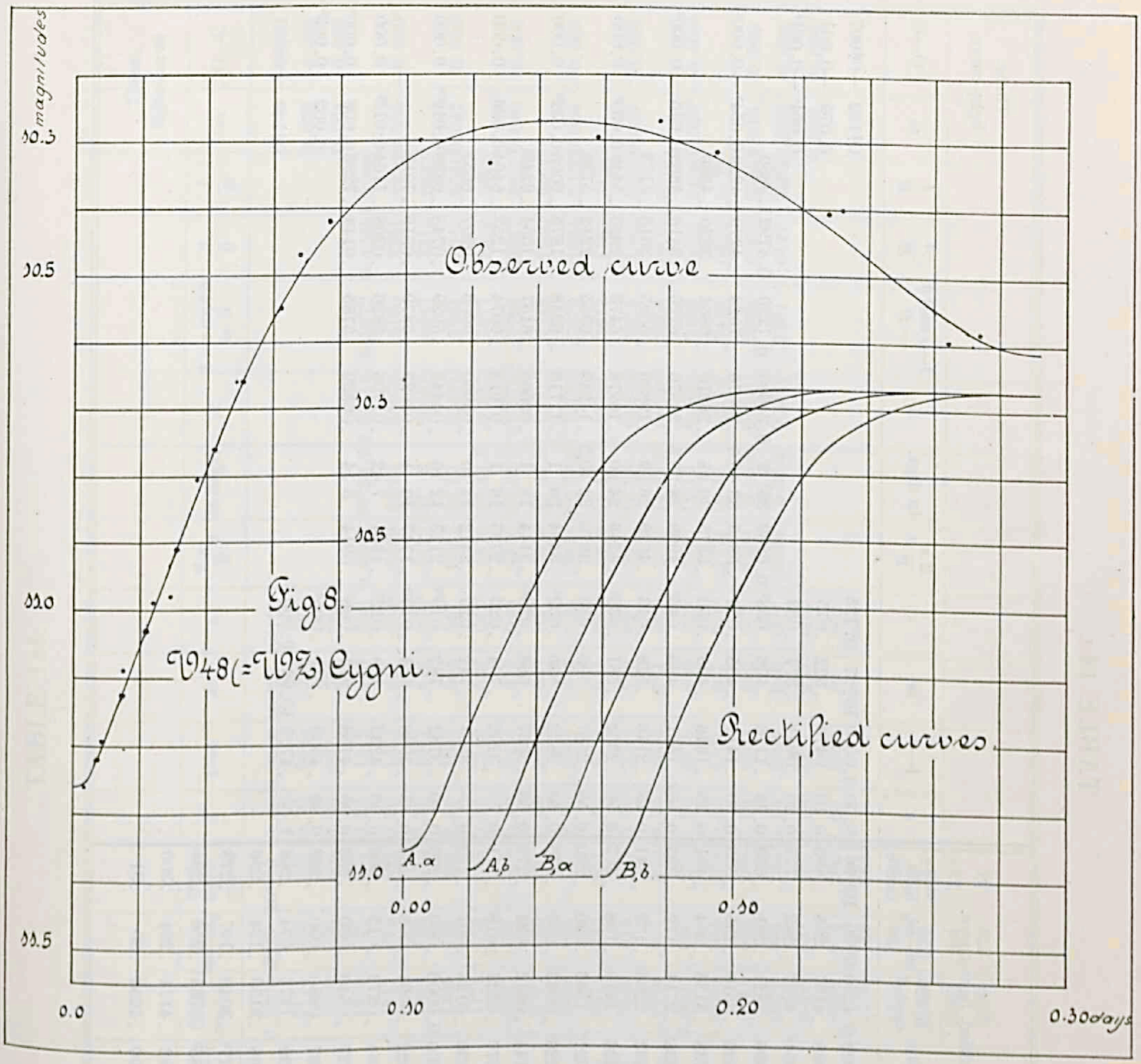
| Depth | | Semi-duration | | | | | | | | | Density | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| prim. | sec. | eclipse | totality | $a_0$ | $L_2$ | $r_1$ | $r_2$ | $k$ | $i$ | $\gamma$ | $\varrho_1$ | $\varrho_2$ | Hypotheses |
| $1^m.48$ | $0^m.10$ | $0^d.075$ | $0^d.008^5$ | 1.00 | 0.744 | 0.338 | 0.199 | 0.59 | 82°49' | 8.4 | 0.241 | 1.182 | U } Pickering. |
| „ | 0.21 | 0.082 | $0.007^5$ | „ | „ | 0.324 | 0.249 | 0.77 | 87 2 | 4.9 | 0.274 | 0.603 | D } Pickering. |
| „ | 0.17 | 0.072 | 0.007 | „ | 0.655 | 0.329 | 0.191 | 0.58 | 82 36 | 5.6 | 0.261 | 1.336 | U } Parkhurst. |
| „ | 0.38 | 0.079 | $0.003^5$ | „ | „ | 0.315 | 0.240 | 0.76 | 85 55 | 3.3 | 0.298 | 0.674 | D } Parkhurst. |

*Remark*: Comparing the observed depth ($0^m.11$) of the secondary minimum with the computed depths, we see that Pickering's light-scale gives by far the better agreement. If we suppose, in the case B,b) the primary minimum to be caused by an annular eclipse, we shall find for $k$ a value (0.47), which is wholly incompatible with eq. (6b), as might have been expected before.

## § 16. V 48 = WZ CYGNI.

| Comparison-stars (1900) | | H.A. 74 | H′ | Normals | | | | Light-curve Fig. 8. | |
|---|---|---|---|---|---|---|---|---|---|
| α | δ | | | | | | | | |
| $20^h48^m22'$ | $+38^\circ31'.8$ | $9^m.19$ | $9^m.22$ | $-0^d.204$ | $10^m.58$ | $+0^d.265$ | $10^m.59^5$ | 0.000 | $11^m.26$ |
| 58 | 22.4 | 10.38 | 10.36 | $0.232^5$ | 10.40 | 0.228 | $10.40^5$ | 0.010 | 11.19 |
| 59 | 28.0 | 11.12 | 11.14 | 0.194 | $10.31^5$ | 0.177 | 10.27 | 0.020 | 11.06 |
| 49 8 | 29.8 | 11.30 | 11.30 | 0.158 | $10.29^5$ | 0.125 | $10.33^5$ | 0.030 | 10.92 |
| | | | | 0.104 | 10.30 | 0.086 | 10.45 | 0.040 | 10.79 |
| | | | | 0.077 | 10.42 | 0.068 | 10.47 | 0.050 | $10.66^5$ |
| | | | | 0.062 | 10.55 | 0.051 | 10.66 | 0.060 | $10.56^5$ |
| | | | | 0.049 | 10.66 | 0.042 | 10.76 | 0.080 | $10.41^5$ |
| | | | | 0.037 | $10.80^5$ | 0.031 | 10.91 | 0.100 | 10.33 |
| | | | | 0.029 | 10.98 | 0.024 | 10.99 | 0.120 | 10.29 |
| | | | | 0.022 | 11.03 | 0.015 | 11.09 | 0.146 | 10.27 |
| | | | | 0.015 | $11.12^5$ | 0.007 | 11.22 | 0.160 | $10.27^5$ |
| | | | | 0.009 | $11.19^5$ | | | 0.180 | $10.29^5$ |
| | | | | 0.003 | 11.26 | | | 0.200 | 10.34 |
| | | | | | | | | 0.220 | 10.40 |
| | | | | | | | | 0.240 | 10.47 |
| | | | | | | | | 0.260 | 10.55 |
| | | | | | | | | 0.280 | 10.60 |
| | | | | | | | | 0.292 | 10.61 |

The brightness between the eclipses is not constant ; this system is a Lyrid.


magnitudes
10.3
10.5
11.0
11.5
Observed curve
Fig. 8.
V48 (= WZ) Cygni.
Rectified curves.
10.5
10.5
11.0
A,α
A,b
B,α
B,b
0.00
0.10
0.0
0.10
0.20
0.30 days

# TABLE 18*a*.

PICKERING. U-hyp.

Determination of $z$ of and the rectified light-curve.

| $\tau$ | $\frac{2\pi}{P}\tau$ | $m$ | $\vartheta$ | $1-L^2$ | $\cos^2\vartheta$ | $0.42^5\cos^2\vartheta$ | $m-m_0$ | rect. light-curve |
|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.000 | 11$^m$26 | 0°0' | | 1.000 | 0.4250 | 0$^m$30 | 10$^m$96 |
| 0.01 | .1075 | .19 | 6 10 | | 0.988 | .4197 | .295 | .895 |
| 0.02 | .215 | .06 | 12 21 | | .954$^5$ | .4057 | .283 | .777 |
| 0.03 | .3225 | 10.92 | 18 29 | | .899 | .3821 | .261 | .659 |
| 0.04 | .430 | .79 | 24 38 | | .826 | .3510 | .235 | .555 |
| 0.05 | .5375 | .66$^5$ | 30 48 | | .738 | .3136 | .204 | .461 |
| 0.06 | .645 | .56$^5$ | 36 57 | | .638$^5$ | .2714 | .172 | .393 |
| 0.07 | .7525 | .48 | 43 7 | | .533 | .2265 | .139 | .341 |
| 0.08 | .860 | .41$^5$ | 49 16 | | .426 | .1810 | .108 | .307 |
| 0.09 | .9675 | .37 | 55 26 | 0.1682 | .321$^5$ | .1366 | .080 | .290 |
| 0.10 | 1.075 | .33 | 61 36 | .1046 | .226$^5$ | .0963 | .055 | .275 |
| 0.11 | .1825 | .30$^6$ | 67 45 | .0641 | .144 | .0612 | .034 | .272 |
| 0.12 | .290 | .28$^9$ | 73 55 | .0344 | .077 | .0327 | .018 | .271 |
| 0.13 | .397 | .27$^7$ | 80 4 | .0128 | .030 | .0128 | .007 | .270 |
| 0.146 | .570 | .27 | 90 | .0000 | .000 | 0.0000 | .000 | .270 |
| 0.16 | .720 | .27$^5$ | 98 33 | .0092 | .022 | .0094 | .005 | .268 |
| 0.17 | .8275 | .28$^3$ | 104 41 | .0237 | .064 | .0272 | .015 | .266 |
| 0.18 | .935 | .29$^6$ | 110 52 | .0460 | .127 | .0540 | .030 | .266 |
| 0.19 | 2.0425 | .31$^6$ | 117 2 | .0812 | .207 | .0880 | .050 | .266 |
| 0.20 | .150 | .34 | 123 11 | .1210 | .299 | .1271 | .074 | .266 |
| 0.22 | .365 | .40 | 135 30 | | .508 | .2159 | .134 | .266 |
| 0.24 | .580 | .47 | 147 50 | | .717$^5$ | .3049 | .197 | .273 |
| 0.26 | .795 | .55 | 160 9 | | .885$^5$ | .3763 | .256 | .294 |
| 0.28 | 3.010 | .60 | 172 28 | | .982 | .4174 | .294 | .306 |
| 0.292 | 3.139 | .61 | 180° | | 1.000 | .4250 | .30 | .31 |

Theor. light-curve

| $a$ | $1-l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | in degr. | $A$ | $1-z\cos^2\vartheta = B$ | $\frac{A}{B}$ | $\frac{1}{B}$ | $\tau_c$ | O—C |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | 10$^m$27 | 0$^m$125 | | | | | | | 0$^d$132 | —0$^d$007 |
| 0.10 | .0470 | .322 | .075 | | | | | | | 0.079 | —0.004 |
| 0.20 | .0941 | .377 | .063 | | | | | | | 0.064$^5$ | —0.001$^5$ |
| 0.25 | .1176 | .406 | .058$^5$ | 0.6289 | 36° 2' | 0.34600 | 0.7220 | 0.4792 | 1.3850 | | |
| 0.30 | .1411 | .435 | .054 | .5805 | 33 15,5 | .30076 | .7028 | .4279 | .4229 | 0.054$^5$ | —0.000$^5$ |
| 0.35 | .1646 | .465 | .050 | .5375 | 30 48 | .26219 | .6864 | .3820 | .4569 | | |
| 0.40 | .1881 | .496 | .046$^5$ | .4999 | 28 38,5 | .22976 | .6726 | .3416 | .4868 | 0.046 | 0.000$^5$ |
| 0.45 | .2116 | .528 | .043 | .4622 | 26 29 | .19886 | .6595 | .3015 | .5163 | | |
| 0.50 | .2352 | .561 | .039$^5$ | .4246 | 24 20 | .16978 | .6473 | .2623 | .5449 | 0.039$^5$ | 0.000 |
| 0.55 | .2587 | .595 | .036 | .3870 | 22 10,5 | .14246 | .6355 | .2242 | .5736 | | |
| 0.60 | .2822 | .630 | .032$^5$ | .3494 | 20 1 | .11716 | .6248 | .1875 | .6005 | 0.032$^5$ | 0.000 |
| 0.65 | .3057 | .666 | .029$^{25}$ | .3144 | 18 1 | .09566 | .6157 | .1554 | .6242 | | |
| 0.70 | .3292 | .704 | .026 | .2795 | 16 1 | .07613 | .6074 | .1253 | .6464 | 0.026 | 0.000 |
| 0.75 | .3527 | .742 | .023 | .2472 | 14 10 | .05990 | .6005 | .0997 | .6653 | | |
| 0.80 | .3762 | .782 | .019$^{75}$ | .2123 | 12 10 | .04442 | .5939 | .0748 | .6838 | 0.019$^{75}$ | 0.000 |
| 0.85 | .3998 | .824 | .016$^{25}$ | .1747 | 10 1 | .03025 | .5879 | .0515 | .7010 | | |
| 0.90 | .4233 | .868 | .012$^5$ | .1344 | 7 42 | .01795 | .5826 | .0308 | .7164 | 0.012$^5$ | 0.000 |
| 0.95 | .4468 | .913 | .008$^5$ | .0914 | 5 14 | .00832 | .5785 | .0144 | .7286 | 0.008 | 0.000$^5$ |
| 0.98 | .4609 | .941 | .005$^5$ | | | | | | | 0.002 | 0.003$^5$ |
| 1.00 | .4703 | 10.96 | 0.000 | | | | | | | | |

# TABLE 18*b*.

| PICKERING D-hyp. | | | | | | | | | | | | | | | | | | | Theor. light-curve | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | $\frac{2\pi}{P}\tau$ | $m$ | $\vartheta$ | $1-L$ | $\cos^2\vartheta$ | $\frac{0.22}{\cos^2\vartheta}$ | $m-m_0$ | rect. light-curve | $\alpha$ | $1-l$ | $m$ | $\tau$ | $\frac{2\pi}{P}\tau$ | in degr. | $A$ | $1-z\cos^2\vartheta = B$ | $\frac{A}{B}$ | $\frac{1}{B}$ | $\tau_c$ | O—C |
| 0.00 | As above | As above | As above | | As above | 0.2200 | $0^m.27$ | $10^m.99$ | 0.00 | 0.0000 | $10^m.27$ | $0^d.125$ | | | | | | | $0^d.146$ | $-0^d.021$ |
| 0.01 | | | | | | .2174 | .265 | .925 | 0.10 | .0485 | .324 | .075 | | | | | | | 0.079 | —0.004 |
| 0.02 | | | | | | .2100 | .256 | .804 | 0.20 | .0970 | .381 | .064 | | | | | | | 0.064 | 0.000 |
| 0.03 | | | | | | .1978 | .239 | .681 | 0.25 | .1212 | .410 | .059 | 0.6342 | 36°20′,5 | 0.35118 | 0.8272 | 0.4245 | 1.2089 | | |
| 0.04 | | | | | | .1817 | .218 | .572 | 0.30 | .1454 | .441 | $.054^5$ | .5859 | 33 34 | .30570 | .8150 | .3751 | .2270 | 0.055 | $-0.000^5$ |
| 0.05 | | | | | | .1624 | .192 | .473 | 0.35 | .1697 | .472 | $.050^5$ | .5429 | 31 6 | .26681 | .8047 | .3315 | .2427 | | |
| 0.06 | | | | | | .1405 | .164 | .401 | 0.40 | .1939 | .504 | .047 | .5052 | 28 57 | .23431 | .7961 | .2942 | .2561 | 0.047 | 0.000 |
| 0.07 | | | | | | .1173 | .135 | .345 | 0.45 | .2182 | .537 | $.043^5$ | .4676 | 26 47,5 | .20317 | .7877 | .2579 | .2695 | | |
| 0.08 | | | | | | .0937 | .107 | .308 | 0.50 | .2424 | .571 | .040 | .4300 | 24 38 | .17373 | .7799 | .2227 | .2822 | 0.040 | 0.000 |
| 0.09 | | | | 0.0880 | | .0707 | .080 | .29 | 0.55 | .2666 | .607 | .037 | .3978 | 22 47,5 | .15007 | .7736 | .1940 | .2927 | | |
| 0.10 | | | | .0538 | | .0498 | .055 | .275 | 0.60 | .2909 | .643 | .034 | .3655 | 20 56,5 | .12775 | .7676 | .1664 | .3028 | 0.034 | 0.000 |
| 0.11 | | | | .0326 | | .0318 | .035 | .271 | 0.65 | .3151 | .681 | $.030^5$ | .3279 | 18 47 | .10368 | .7613 | .1363 | .3135 | | |
| 0.12 | | | | .0174 | | .0169 | .018 | .271 | 0.70 | .3394 | .720 | .027 | .2902 | 16 38 | .08200 | .7555 | .1085 | .3236 | 0.027 | 0.000 |
| 0.13 | | | | .0064 | | .0066 | .007 | .270 | 0.75 | .3636 | .761 | $.023^5$ | .2526 | 14 28,5 | .06248 | .7503 | .0833 | .3328 | | |
| 0.146 | | | | .0000 | | 0.0000 | .000 | .270 | 0.80 | .3878 | .803 | .020 | .2150 | 12 19 | .04550 | .7458 | .0610 | .3408 | $0.020^5$ | $-0.000^5$ |
| 0.16 | | | | .0046 | | .0048 | .005 | .270 | 0.85 | .4121 | .847 | $.016^5$ | .1774 | 10 10 | .03116 | .7419 | .0420 | .3479 | | |
| 0.17 | | | | .0119 | | .0141 | .015 | .268 | 0.90 | .4363 | .892 | .013 | .1398 | 8 0,5 | .01941 | .7388 | .0263 | .3535 | 0.013 | 0.000 |
| 0.18 | | | | .0237 | | .0279 | .031 | .265 | 0.95 | .4606 | .940 | .009 | .0968 | 5 32,5 | .00933 | .7361 | .0127 | .3585 | 0.007 | 0.002 |
| 0.19 | | | | .0415 | | .0455 | .051 | .265 | 0.98 | .4751 | .970 | .006 | | | | | | | 0.000 | 0.006 |
| 0.20 | | | | .0624 | | .0658 | .074 | .266 | 1.00 | .4848 | .99 | .000 | | | | | | | | |
| 0.22 | | | | | | .1118 | .129 | .269 | | | | | | | | | | | | |
| 0.24 | | | | | | .1578 | .186 | .284 | | | | | | | | | | | | |
| 0.26 | | | | | | .1949 | .236 | .314 | | | | | | | | | | | | |
| 0.28 | | | | | | .2160 | .264 | .336 | | | | | | | | | | | | |
| 0.292 | | | | | | .2200 | 0.270 | .34 | | | | | | | | | | | | |

# V 48 = WZ CYGNI.

$$P = 0^d\!.5845\,; \quad \frac{2\pi}{P} = 10.75.$$

**A. Pickering**

**a**

In the manner described (p. 31), we find :

$z$: (Tables 18$a$ and $b$) — $z = 0^m\!.425$

Rectified Light-curve :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0^d\!.000$ | $10^m\!.96$ | $0^d\!.030$ | $10^m\!.66$ | $0^d\!.060$ | $10^m\!.39$ | $0^d\!.090$ | $10^m\!.29$ |
| .010 | $.89^5$ | .040 | $.55^5$ | .070 | .34 | .100 | $.27^5$ |
| .020 | $.77^{75}$ | .050 | .46 | .080 | $.30^5$ | .125 | .27 |
| | | | | | | .130 | .27 |

Depth of primary minimum $0^m\!.69$; of secondary minimum $0^m\!.04$.

Determination of $k$:

$\overline{A_1}=0.25122$ $\overline{B_1}=0.6818$ $\frac{\overline{A_1}}{\overline{B_1}}=0.3658$ $\frac{1}{\overline{B_1}}=1.4688$ $\alpha_1=1.1030$ $R_1=-0.36067$

$\overline{A_2}=0.09826$ $\overline{B_2}=0.6168$ $\frac{\overline{A_2}}{\overline{B_2}}=0.1584$ $\frac{1}{\overline{B_2}}=1.6220$ $\alpha_2=1.4636$ $R_2=\phantom{-}0.20105$

$\overline{A_3}=0.02524$ $\overline{B_3}=0.5857$ $\frac{\overline{A_3}}{\overline{B_3}}=0.0428$ $\frac{1}{\overline{B_3}}=1.7075$ $\alpha_3=1.6647$ $R_3=-0.000023$

$\frac{A_1}{B_1}=0.3684$ $\frac{1}{B_1}=1.467$ $\overline{A_1}-\overline{A_2}=0.15296$ $\overline{B_3}\,(\overline{A_2}-\overline{A_1})=-0.08959$

$\frac{A_2}{B_2}=0.1593$ $\frac{1}{B_2}=1.621$ $\overline{A_2}-\overline{A_3}=0.07302$ $\overline{B_1}\,(\overline{A_3}-\overline{A_2})=-0.04978$

$\frac{A_3}{B_3}=0.0431$ $\frac{1}{B_3}=1.707$ $\overline{A_3}-\overline{A_1}=0.22598$ $\overline{B_2}\,(\overline{A_1}-\overline{A_3})=\phantom{-}0.13938$

$\{\ldots\ldots\ldots\}=\phantom{-}0.000011$

$$\alpha_0 = 0.4703 + \frac{0.0362}{k^2}\ (5a), \quad \text{or:} \quad \alpha_0 = 0.0362 + \frac{0.4703}{k^2}\ (5b).$$

$\alpha_0 = 1.00$; eq. (5*): $k = 0.688$; eq. (8*): $k = 0.688$; eq. (6*): $k = 0.690$; eq. (7*); $k = 0.691$.

It appears therefore, that the approximative equations (8*), (7*) and (6*) give practically the same value for $k$ as eq. (5*). As is remarked on p. 52 we shall use eq. (8*) in future.

$\alpha_0 = 1.00$; eq. (8*): $k = 0.688$; (5$a$): $k = 0.261$; (5$b$): $k = 0.699$

$\alpha_0 = 0.90$; eq. (8*): $k = 0.726$; (5$b$): $k = 0.738$

$\alpha_0 = 0.85$; eq. (8*): $k = 0.757$; (5$b$): $k = 0.760$

$\alpha_0 = 0.84$; eq. (8*): $k = 0.765$; (5$b$): $k = 0.765$

The large star is eclipsed at primary minimum.

**b**

According to p. 36:

$Z = 0.22$; therefore $z = 0.266$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0^d\!.000$ | $10^m\!.99$ | $0^d\!.030$ | $10^m\!.68^{25}$ | $0^d\!.060$ | $10^m\!.40$ | $0^d\!.090$ | $10^m\!.28^5$ |
| .010 | $.92^5$ | .040 | $.57^5$ | .070 | $.34^{25}$ | .100 | $.27^5$ |
| .020 | .80 | .050 | $.47^5$ | .080 | $.30^{75}$ | .125 | .27 |
| | | | | | | .130 | .27 |

Depth of primary minimum $0^m\!.72$; of secondary minimum $0^m\!.07$.

$\overline{A_1}=0.25582$ $\overline{B_1}=0.8018$ $\frac{\overline{A_1}}{\overline{B_1}}=0.3176^5$ $\frac{1}{\overline{B_1}}=1.2477$ $\alpha_1=0.9300$ $R_1=-0.24536$

$\overline{A_2}=0.10520$ $\overline{B_2}=0.7616^5$ $\frac{\overline{A_2}}{\overline{B_2}}=0.1377$ $\frac{1}{\overline{B_2}}=1.3131$ $\alpha_2=1.1754$ $R_2=\phantom{-}0.13931$

$\overline{A_3}=0.02635$ $\overline{B_3}=0.7406^5$ $\frac{\overline{A_3}}{\overline{B_3}}=0.0355$ $\frac{1}{\overline{B_3}}=1.3502$ $\alpha_3=1.3147$ $R_3=\phantom{-}0.000006$

$$\alpha_0' = 0.4848 + \frac{0.0624}{Q(k,\alpha_0')}\ (6a), \quad \text{or:} \quad \alpha_0' = 0.0624 + \frac{0.4848}{Q(k,\alpha_0')}\ (6b).$$

$\alpha_0' = 1.00$; eq. (6$a$): $k = 0.33$; eq. (6$b$): $k = 0.655$; eq. (8*), case E$_l$: $k = 0.89$; case E$_s$: $k = 0.60$

$\alpha_0' = 0.80$; eq. (8*), case E$_l$: $k > 1.00$.

The large star is eclipsed at primary minimum.

$\alpha_0' = 0.75$; (6$b$): $Q(k, \alpha_0') = 0.705$; $k = 0.828$; $\alpha_0'' = 0.667$; eq. (8*): $k = 0.815$

$\alpha_0' = 0.74^5$; (6$b$): $Q(k, \alpha_0') = 0.711$; $k = 0.831$; $\alpha_0'' = 0.663$; eq. (8*): $k = 0.830$

$P = 0^d5845; \quad \frac{2\pi}{P} = 10.75.$

## A. Pickering

### a

In the manner described (p. 31), we find:

z: (Tables 18a and b) — $z = 0^m425$

Rectified Light-curve:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0d000 | 10m96 | 0d030 | 10m66 | 0d060 | 10m39 | 0d090 | 10m29 |
| .010 | .89⁵ | .040 | .55⁵ | .070 | .34 | .100 | .27⁵ |
| .020 | .77⁷⁵ | .050 | .46 | .080 | .30⁵ | .125 | .27 |
| | | | | | | .130 | .27 |

Determination of k:

Depth of primary minimum $0^m69$; of secondary minimum $0^m04$.

$\overline{A_1}=0.25122$ $\overline{B_1}=0.6818$ $\frac{\overline{A_1}}{\overline{B_1}}=0.3658$ $\frac{1}{\overline{B_1}}=1.4688$ $a_1=1.1030$ $R_1=-0.36067$

$\overline{A_2}=0.09826$ $\overline{B_2}=0.6168$ $\frac{\overline{A_2}}{\overline{B_2}}=0.1584$ $\frac{1}{\overline{B_2}}=1.6220$ $a_2=1.4636$ $R_2=\ \ 0.20105$

$\overline{A_3}=0.02524$ $\overline{B_3}=0.5857$ $\frac{\overline{A_3}}{\overline{B_3}}=0.0428$ $\frac{1}{\overline{B_3}}=1.7075$ $a_3=1.6647$ $R_3=-0.000023$

$\frac{A_1}{B_1}=0.3684$ $\frac{1}{B_1}=1.467$ $\overline{A_1}-\overline{A_2}=0.15296$ $\overline{B_3}\,(\overline{A_2}-\overline{A_1})=-0.08959$

$\frac{A_2}{B_2}=0.1593$ $\frac{1}{B_2}=1.621$ $\overline{A_2}-\overline{A_3}=0.07302$ $\overline{B_1}\,(\overline{A_3}-\overline{A_2})=-0.04978$

$\frac{A_3}{B_3}=0.0431$ $\frac{1}{B_3}=1.707$ $\overline{A_3}-\overline{A_1}=0.22598$ $\overline{B_2}\,(\overline{A_1}-\overline{A_3})=\ \ 0.13938$

$\{\ldots\ldots\ldots\}=\ \ 0.000011$

$$a_0 = 0.4703 + \frac{0.0362}{k^2}\ (5a), \quad \text{or:} \quad a_0 = 0.0362 + \frac{0.4703}{k^2}\ (5b).$$

$a_0 = 1.00$; eq. (5*): $k = 0.688$; eq. (8*): $k = 0.688$; eq. (6*): $k = 0.690$; eq. (7*); $k = 0.691$.

It appears therefore, that the approximative equations (8*), (7*) and (6*) give practically the same value for $k$ as eq. (5*). As is remarked on p. 52 we shall use eq. (8*) in future.

$a_0 = 1.00$; eq. (8*): $k = 0.688$; (5a): $k = 0.261$; (5b): $k = 0.699$
$a_0 = 0.90$; eq. (8*): $k = 0.726$; (5b): $k = 0.738$
$a_0 = 0.85$; eq. (8*): $k = 0.757$; (5b): $k = 0.760$
$a_0 = 0.84$; eq. (8*): $k = 0.765$: (5b): $k = 0.765$

The large star is eclipsed at primary minimum.

*Adopted:* $a_0 = 0.84$; $k = 0.765$.

Eq. (4*): $1.1030 \sin^2 i + 1.3106\, a_1^2 = 1.4688$ $\quad \sin^2 i = 0.9546 \quad i = 77°42'$

$1.6647 \sin^2 i + 0.3731\, a_1^2 = 1.7075$ $\quad a_1^2 = 0.3173$ $a_1 = 0.563$ $a_2 = ka_1 = 0.431$

$z = \varepsilon^2 \sin^2 i$, whence $\varepsilon = 0.67$

Least apparent distance of centers: $b_1^2 = a_1^2\,(1-\varepsilon^2)$; $b_1 = 0.418$; $b_2 = kb_1 = 0.320$

$\cos i = 0.213$

$L_1$ and $L_2$: $L_1 = \frac{1-\lambda_{pr}}{a_0 k^2} = \frac{0.4703}{0.4914} = 0.958$; $\quad L_2 = \frac{1-\lambda_{sec}}{a_0} = 0.043.$

Control $L_1 + L_2 = 1.00$.

$\frac{L_2}{L_1} = 0.045$, therefore $\gamma = 0.076$.

$\varrho_1$ and $\varrho_2$: $\varrho_1 = 0.260$; $\varrho_2 = 0.582$.

### b

According to p. 36:

$Z = 0.22$; therefore $z = 0.266$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0d000 | 10m99 | 0d030 | 10m68²⁵ | 0d060 | 10m40 | 0d090 | 10m28⁵ |
| .010 | .92⁵ | .040 | .57⁵ | .070 | .34²⁵ | .100 | .27⁵ |
| .020 | .80 | .050 | .47⁵ | .080 | .30⁷⁵ | .125 | .27 |
| | | | | | | .130 | .27 |

Depth of primary minimum $0^m72$; of secondary minimum $0^m07$.

$\overline{A_1}=0.25582$ $\overline{B_1}=0.8018$ $\frac{\overline{A_1}}{\overline{B_1}}=0.3176^5$ $\frac{1}{\overline{B_1}}=1.2477$ $a_1=0.9300$ $R_1=-0.24536$

$\overline{A_2}=0.10520$ $\overline{B_2}=0.7616^5$ $\frac{\overline{A_2}}{\overline{B_2}}=0.1377$ $\frac{1}{\overline{B_2}}=1.3131$ $a_2=1.1754$ $R_2=\ \ 0.13931$

$\overline{A_3}=0.02635$ $\overline{B_3}=0.7406^5$ $\frac{\overline{A_3}}{\overline{B_3}}=0.0355$ $\frac{1}{\overline{B_3}}=1.3502$ $a_3=1.3147$ $R_3=\ \ 0.000006$

$$a_0' = 0.4848 + \frac{0.0624}{Q\,(k, a_0')}\ (6a), \quad \text{or:} \quad a_0' = 0.0624 + \frac{0.4848}{Q\,(k, a_0')}\ (6b).$$

$a_0' = 1.00$; eq. (6a): $k = 0.33$; eq. (6b): $k = 0.655$; eq. (8*), case $E_t$: $k = 0.89$; case $E_s$: $k = 0.60$

$a_0' = 0.80$; eq. (8*), case $E_t$: $k > 1.00$.

The large star is eclipsed at primary minimum.

$a_0' = 0.75$; (6b): $Q\,(k, a_0') = 0.705$; $k = 0.828$; $a_0'' = 0.667$; eq. (8*): $k = 0.815$

$a_0' = 0.74^5$; (6b): $Q\,(k, a_0') = 0.711$; $k = 0.831$; $a_0'' = 0.663$; eq. (8*): $k = 0.830$

$a_0 = 0.745$; $k = 0.83$.

$0.9300 \sin^2 i + 1.3355\, a_1^2 = 1.2477$ $\quad \sin^2 i = 0.9080 \quad i = 72°29'$

$1.3147 \sin^2 i + 0.5181\, a_1^2 = 1.3502$ $\quad r_1^2 = 0.3019$ $a_1 = 0.549$ $a_2 = ka_1 = 0.431$

$\varepsilon^2 = z \cos^2 i$; whence $\varepsilon = 0.54$.

$b_1^2 = a_1^2\,(1-\varepsilon^2)$; $b_1 = 0.463$ $b_2 = k\,b_1 = 0.384$

$\cos i = 0.301$

$L_1 = \frac{1-\lambda_{pr}}{a_0'\,Q\,(k, a_0')} = 0.916^5$ $\quad L_2 = \frac{1-\lambda_{sec}}{a_0'} = 0.084$

Control $L_1 + L_2 = 1.00$

$\frac{L_2}{L_1} = 0.092$; mean $\gamma = 0.133$.

$\varrho_1 = 0.191$ $\quad \varrho_2 = 0.265$.

## B. (Parkhurst)

### a

z: Rectified Light-curve: $z = 0.34$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0d000 | 10m97 | 0d030 | 10m6675 | 0d060 | 10m39⁵ | 0d090 | 10m2875 |
| .010 | .90⁵ | .040 | .56 | .070 | .34 | .100 | .27⁵ |
| .020 | .78⁵ | .050 | .46⁵ | .080 | .30⁵ | .125 | .27 |
| | | | | | | .130 | .27 |

Determination of k:

Depth of primary minimum $0^m70$; of secondary minimum $0^m05$.

$\overline{A_1} = 0.25853$ $\quad \overline{B_1} = 0.7411$ $\quad \overline{B_3}\,(\overline{A_1}-\overline{A_2}) = \ \ 0.09619$

$\overline{A_2} = 0.09451$ $\quad \overline{B_2} = 0.6921$ $\quad \overline{B_1}\,(\overline{A_3}-\overline{A_2}) = -0.05289$

$\overline{A_3} = 0.02314$ $\quad \overline{B_3} = 0.6679$ $\quad [\overline{B_2}\,(\overline{A_1}-\overline{A_3}) = \ \ 0.14908]$

$$a_0 = 0.0353 + \frac{0.3952}{k^2}\ (5a), \text{ or: } a_0 = 0.3952 + \frac{0.0353}{k^2}\ (5b)$$

$a_0 = 1.00$; eq. (7*): $k = 0.70$; eq. (5a): $k = 0.24$; eq. (5b): $k = 0.64$

$a_0 = 0.65$; eq. (7*): $k > 1.00$; eq. (5a): $k = 0.37$; eq. (5b): $k = 0.80$

Starting from Parkhurst's light-scale we, therefore, cannot find a set of elements, yielding the observed light-curve.

### b

$Z = 0.17$; therefore $z = 0.208$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0d000 | 11m00 | 0d030 | 10m69 | 0d060 | 10m40⁵ | 0d090 | 10m29 |
| .010 | 10.93⁵ | .040 | .58 | .070 | .34⁷⁵ | .100 | .27⁵ |
| .020 | .81⁵ | .050 | .48 | .080 | .31 | .125 | .27 |
| | | | | | | .130 | .27 |

Depth of primary minimum $0^m73$; of secondary minimum $0^m08$.

$\overline{A_1} = 0.24426$ $\quad \overline{B_1} = 0.8428$ $\quad \overline{B_3}\,(\overline{A_1}-\overline{A_2}) = \ \ 0.11686$

$\overline{A_2} = 0.09765$ $\quad \overline{B_2} = 0.8123$ $\quad \overline{B_1}\,(\overline{A_3}-\overline{A_1}) = -0.06146$

$\overline{A_3} = 0.02472$ $\quad \overline{B_3} = 0.7971$ $\quad [\overline{B_2}\,(\overline{A_1}-\overline{A_3}) = \ \ 0.17833]$

$$a_0' = 0.4081 + \frac{0.0558}{Q\,(k, a_0')}\ (6a), \text{ or: } a_0' = 0.0558 + \frac{0.4081}{Q\,(k, a_0')}\ (6b)$$

$a_0' = 1.00$; eq. (6a): $k = 0.30$; eq. (6b): $k = 0.60$; eq. (7*) case $E_t$: $k = 0.94$; case $E_s$: $k = 0.72$

$a_0' = 0.80$; eq. (6a): $k = 0.40$; eq. (6b): $k = 0.73$; eq. (7*) in both cases $k > 1.00$.

Starting from Parkhurst's light-scale we, therefore, cannot find a set of elements, yielding the observed light-curve.

## SUMMARY.

| Observed Depth | | Rectified Depth | | Semi-duration | | | | | | | | | | Density | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| prim. | sec. | prim. | sec. | eclipse | $a_0\,(a_0')$ | $L_1$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ | $k$ | $i$ | $\gamma$ | $\varrho_1$ | $\varrho_2$ | Hypotheses. |
| 0m99 | 0m34 | 0m69 | 0m04 | 0d132 | 0.84 | 0.958 | 0.563 | 0.431 | 0.418 | 0.320 | 0.765 | 77°42' | 0.076 | 0.260 | 0.582 | U } Pickering. |
| ,, | ,, | 0.72 | 0.07 | 0.146 | 0.74⁵ | 0.916 | 0.549 | 0.456 | 0.463 | 0.384 | 0.83 | 72 29 | 0.133 | 0.191 | 0.265 | D } Pickering. |
| ,, | ,, | 0.70 | 0.05 | No set of elements. | | | | | | | | | | | | U } Parkhurst. |
| ,, | ,, | 0.73 | 0.08 | ,, ,, ,, ,, | | | | | | | | | | | | D } Parkhurst. |

# TABLES

*Loss of Light corresponding to an Increase $\Delta m$ in Stellar Magnitude.*

TABLE A

($\mu = 2.512$ ; $\log \mu = 0.4$)

| $\Delta m$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| 0.0.... | 0.0000 | 0.0183 | 0.0362 | 0.0538 | 0.0710 |
| .1.... | .0880 | .1046 | .1210 | .1370 | .1528 |
| .2.... | .1682 | .1834 | .1983 | .2130 | .2273 |
| .3.... | .2414 | .2553 | .2689 | .2822 | .2953 |
| .4.... | .3082 | .3208 | .3332 | .3454 | .3573 |
| .5.... | .3690 | .3806 | .3919 | .4030 | .4139 |
| .6.... | .4246 | .4351 | .4454 | .4555 | .4654 |
| .7.... | .4752 | .4848 | .4942 | .5034 | .5125 |
| .8.... | .5214 | .5301 | .5387 | .5471 | .5554 |
| .9.... | .5635 | .5715 | .5793 | .5870 | .5945 |
| 1.0.... | .6019 | .6092 | .6163 | .6233 | .6302 |
| .1.... | .6369 | .6435 | .6501 | .6564 | .6627 |
| .2.... | .6689 | .6749 | .6808 | .6867 | .6924 |
| .3.... | .6980 | .7035 | .7089 | .7142 | .7195 |
| .4.... | .7246 | .7296 | .7345 | .7394 | .7441 |
| .5.... | .7488 | .7534 | .7579 | .7623 | .7667 |
| .6.... | .7709 | .7751 | .7792 | .7832 | .7872 |
| .7.... | .7911 | .7949 | .7986 | .8023 | .8059 |
| .8.... | .8095 | .8129 | .8163 | .8197 | .8230 |
| .9.... | .8262 | .8294 | .8325 | .8356 | .8386 |
| 2.0.... | .8415 | .8444 | .8472 | .8500 | .8528 |
| .1.... | .8555 | .8581 | .8607 | .8632 | .8657 |
| .2.... | .8682 | .8706 | .8729 | .8753 | .8775 |
| .3.... | .8798 | .8820 | .8841 | .8862 | .8883 |
| .4.... | .8904 | .8924 | .8943 | .8962 | .8981 |
| .5.... | .9000 | .9018 | .9036 | .9054 | .9071 |

TABLE B

($\mu = 2.05$; $\log \mu = 0.312$)

| $\Delta m$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| 0.0.... | 0.0000 | 0.0143 | 0.0283 | 0.0422 | 0.0558 |
| .1.... | .0693 | .0826 | .0957 | .1086 | .1213 |
| .2.... | .1338 | .1462 | .1584 | .1704 | .1822 |
| .3.... | .1939 | .2054 | .2167 | .2279 | .2389 |
| .4.... | .2498 | .2605 | .2710 | .2814 | .2917 |
| .5.... | .3018 | .3117 | .3216 | .3312 | .3408 |
| .6.... | .3502 | .3594 | .3686 | .3776 | .3865 |
| .7.... | .3952 | .4038 | .4124 | .4207 | .4290 |
| .8.... | .4371 | .4452 | .4531 | .4609 | .4686 |
| .9.... | .4762 | .4836 | .4910 | .4983 | .5054 |
| 1.0.... | .5125 | .5194 | .5263 | .5330 | .5397 |
| .1.... | .5463 | .5527 | .5591 | .5654 | .5716 |
| .2.... | .5777 | .5837 | .5897 | .5955 | .6013 |
| .3.... | .6070 | .6126 | .6181 | .6236 | .6289 |
| .4.... | .6342 | .6395 | .6446 | .6497 | .6547 |
| .5.... | .6596 | .6644 | .6692 | .6740 | .6786 |
| .6.... | .6832 | .6877 | .6922 | .6966 | .7009 |
| .7.... | .7051 | .7094 | .7135 | .7176 | .7216 |
| .8.... | .7256 | .7295 | .7334 | .7372 | .7409 |
| .9.... | .7446 | .7483 | .7518 | .7554 | .7589 |
| 2.0.... | .7623 | .7657 | .7690 | .7723 | .7756 |
| .1.... | .7788 | .7819 | .7851 | .7881 | .7911 |
| .2.... | .7941 | .7971 | .8000 | .8028 | .8056 |
| .3.... | .8084 | .8111 | .8138 | .8165 | .8191 |
| .4.... | .8217 | .8242 | .8267 | .8292 | .8316 |
| .5.... | .8340 | .8364 | .8387 | .8410 | .8433 |
| .6.... | .8455 | .8477 | .8499 | .8521 | .8542 |
| .7.... | .8563 | .8583 | .8603 | .8623 | .8643 |
| .8.... | .8662 | .8681 | .8700 | .8719 | .8737 |
| .9.... | .8755 | .8773 | .8790 | .8807 | .8824 |
| 3.0.... | .8841 | .8858 | .8874 | .8890 | .8906 |
| .1.... | .8922 | .8937 | .8952 | .8967 | .8982 |
| .2.... | .8996 | .9011 | .9025 | .9039 | .9052 |

A. For values of $\Delta m$ greater than $2^{\mathrm{m}}.50$ the loss of light is 0,9000 plus $\frac{1}{10}$ of the loss of light corresponding to $\Delta m - 2^{\mathrm{m}}.50$.

B. For values of $\Delta m$ greater than $3^{\mathrm{m}}.205$ the loss of light is 0,9000 plus $\frac{1}{10}$ of the loss of light corresponding to $\Delta m - 3^{\mathrm{m}}.205$.

TABLE C

*Values of* $Q(k, \alpha_0')$.

| $\alpha_0'$ \ $k$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| 0.00.... | 1.00 | 0.769 | 0.573 | 0.410 | 0.279 | 0.177 | 0.102 |
| 0.10.... | " | .777 | .585 | .426 | .292 | .188 | .108 |
| 0.20.... | " | .786 | .596 | .438 | .302 | .195 | .113 |
| 0.30.... | " | .796 | .605 | .446 | .310 | .200 | .116 |
| 0.40.... | " | .805 | .614 | .453 | .316 | .205 | .119 |
| 0.50.... | " | .814 | .624 | .461 | .321 | .210 | .123 |
| 0.60.... | " | .823 | .634 | .470 | .328 | .215 | .126 |
| 0.70.... | " | .832 | .646 | .483 | .339 | .222 | .131 |
| 0.80.... | " | .844 | .661 | .499 | .352 | .232 | .137 |
| 0.90.... | " | .861 | .685 | .522 | .371 | .247 | .147 |
| 0.95.... | " | .872 | .704 | .538 | .384 | .256 | .153 |
| 0.98.... | " | .880 | .717 | .553 | .398 | .267 | .160 |
| 0.99.... | " | .890 | .728 | .563 | .405 | .273 | .164 |
| 1.00.... | " | .904 | .750 | .587 | .427 | .289 | .175 |
| 1+x.... | " | .917 | .784 | .636 | .488 | .351 | .230 |

# TABLE I.

## *Values of p.* (U-hyp.).

| $\alpha$ \ $k$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.00.... | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 |
| 0.01.... | 0.919 | 0.921 | 0.922 | 0.924 | 0.925 | 0.927 | 0.929 | 0.930 | 0.932 | 0.934 | 0.935 |
| 0.02.... | .868 | .871 | .873 | .876 | .879 | .881 | .884 | .887 | .890 | .892 | .895 |
| 0.05.... | .755 | .759 | .764 | .769 | .774 | .779 | .785 | .790 | .795 | .800 | .805 |
| 0.10.... | .610 | .618 | .624 | .631 | .638 | .645 | .653 | .661 | .670 | .678 | .687 |
| 0.15.... | .488 | .496 | .504 | .513 | .523 | .533 | .544 | .554 | .565 | .576 | .585 |
| 0.20.... | .374 | .388 | .398 | .408 | .419 | .430 | .443 | .456 | .469 | .481 | .492 |
| 0.25.... | .267 | .284 | .297 | .310 | .322 | .335 | .348 | .363 | .378 | .391 | .405 |
| 0.30.... | .168 | .186 | .200 | .216 | .230 | .244 | .258 | .272 | .288 | .303 | .321 |
| 0.35.... | +0.075 | .094 | .110 | .127 | .143 | .160 | .175 | .190 | .207 | .222 | .239 |
| 0.40.... | −0.015 | +0.005 | +0.024 | +0.041 | +0.059 | +0.077 | .094 | .109 | .126 | .143 | .159 |
| 0.45.... | .106 | −0.081 | −0.061 | −0.042 | −0.023 | −0.004 | +0.013 | +0.028 | +0.045 | +0.062 | +0.079 |
| 0.50.... | .194 | .166 | .145 | .124 | .103 | .084 | −0.067 | −0.051 | −0.034 | −0.017 | −0.000 |
| 0.55.... | .280 | .250 | .226 | .204 | .184 | .165 | .148 | .131 | .113 | .096 | .079 |
| 0.60.... | .364 | .332 | .306 | .284 | .263 | .244 | .226 | .209 | .192 | .175 | .159 |
| 0.65.... | .447 | .413 | .386 | .363 | .343 | .323 | .305 | .288 | .271 | .255 | .239 |
| 0.70.... | .528 | .492 | .465 | .441 | .420 | .401 | .383 | .367 | .350 | .336 | .321 |
| 0.75.... | .607 | .571 | .544 | .520 | .498 | .481 | .463 | .448 | .432 | .419 | .405 |
| 0.80.... | .686 | .649 | .622 | .600 | .580 | .563 | .546 | .532 | .517 | .504 | .492 |
| 0.85.... | .765 | .728 | .701 | .680 | .663 | .648 | .633 | .620 | .607 | .596 | .585 |
| 0.90.... | .843 | .807 | .783 | .764 | .749 | .736 | .725 | .715 | .705 | .696 | .687 |
| 0.95.... | .922 | .890 | .872 | .858 | .847 | .838 | .830 | .823 | .817 | .811 | .805 |
| 0.98.... | .967 | .945 | .935 | .928 | .922 | .915 | .910 | .905 | .900 | .896 | .892 |
| 0.99.... | 0.983 | 0.967 | 0.960 | 0.955 | 0.951 | 0.948 | 0.945 | 0.942 | 0.939 | 0.937 | 0.934 |
| 1.00.... | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 | −1.000 |
| $\overline{p_1}$.... | +0.0325 | +0.0537 | +0.0708 | +0.0880 | +0.1047 | +0.1213 | +0.1368 | +0.1518 | +0.1683 | +0.1840 | +0.2005 |
| $\overline{p_2}$.... | −0.4452 | −0.4116 | −0.3854 | −0.3624 | −0.3416 | −0.3228 | −0.3050 | −0.2886 | −0.2716 | −0.2562 | −0.2406 |
| $\overline{p_3}$.... | −0.8040 | −0.7685 | −0.7445 | −0.7255 | −0.7098 | −0.6962 | −0.6835 | −0.6725 | −0.6615 | −0.6518 | −0.6422 |
| $\overline{p_1^2}$.... | 0.0257 | 0.0264 | 0.0276 | 0.0296 | 0.0319 | 0.0350 | 0.0386 | 0.0429 | 0.0480 | 0.0531 | 0.0593 |
| $\overline{p_2^2}$.... | 0.2116 | 0.1823 | 0.1612 | 0.1438 | 0.1290 | 0.1166 | 0.1054 | 0.0958 | 0.0864 | 0.0787 | 0.0711 |
| $\overline{p_3^2}$.... | 0.6541 | 0.5986 | 0.5630 | 0.5356 | 0.5136 | 0.4952 | 0.4783 | 0.4640 | 0.4500 | 0.4378 | 0.4261 |

TABLE I*a* ($\alpha_0 = 0.90$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1023 | +0.1217 | +0.1377 | +0.1535 | +0.1688 | +0.1847 | +0.1998 | +0.2145 | +0.2307 | +0.2457 | +0.2613 |
| $\overline{p_2}$ | —0.3378 | —0.3062 | —0.2818 | —0.2594 | —0.2390 | —0.2196 | —0.2022 | —0.1856 | —0.1680 | —0.1518 | —0.1386 |
| $\overline{p_3}$ | —0.6665 | —0.6298 | —0.6028 | —0.5802 | —0.5608 | —0.5440 | —0.5270 | —0.5130 | —0.4982 | —0.4852 | —0.4730 |
| $\overline{p_1^2}$ | 0.0314 | 0.0351 | 0.0384 | 0.0423 | 0.0466 | 0.0515 | 0.0567 | 0.0628 | 0.0699 | 0.0766 | 0.0843 |
| $\overline{p_2^2}$ | 0.1254 | 0.1044 | 0.0898 | 0.0776 | 0.0674 | 0.0584 | 0.0509 | 0.0445 | 0.0383 | 0.0334 | 0.0286 |
| $\overline{p_3^2}$ | 0.4506 | 0.4029 | 0.3696 | 0.3431 | 0.3212 | 0.3024 | 0.2848 | 0.2706 | 0.2560 | 0.2431 | 0.2318 |

TABLE I*b* ($\alpha_0 = 0.80$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1738 | +0.1913 | +0.2058 | +0.2205 | +0.2345 | +0.2490 | +0.2637 | +0.2780 | +0.2937 | +0.3080 | +0.3232 |
| $\overline{p_2}$ | —0.2272 | —0.1988 | —0.1760 | —0.1552 | —0.1352 | —0.1162 | —0.0990 | —0.0828 | —0.0656 | —0.0486 | —0.0318 |
| $\overline{p_3}$ | —0.5272 | —0.4920 | —0.4650 | —0.4415 | —0.4202 | —0.4018 | —0.3840 | —0.3678 | —0.3512 | —0.3370 | —0.3220 |
| $\overline{p_1^2}$ | 0.0477 | 0.0538 | 0.0588 | 0.0643 | 0.0700 | 0.0764 | 0.0838 | 0.0913 | 0.1000 | 0.1081 | 0.1170 |
| $\overline{p_2^2}$ | 0.0611 | 0.0485 | 0.0395 | 0.0324 | 0.0264 | 0.0218 | 0.0180 | 0.0150 | 0.0124 | 0.0104 | 0.0090 |
| $\overline{p_3^2}$ | 0.2832 | 0.2472 | 0.2214 | 0.1999 | 0.1814 | 0.1664 | 0.1524 | 0.1402 | 0.1288 | 0.1189 | 0.1092 |

TABLE I*c* ($\alpha_0 = 0.70$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.2475 | +0.2633 | +0.2763 | +0.2897 | +0.3028 | +0.3160 | +0.3297 | +0.3432 | +0.3583 | +0.3717 | +0.3862 |
| $\overline{p_2}$ | —0.1134 | —0.0888 | —0.0686 | —0.0494 | —0.0304 | —0.0114 | +0.0052 | +0.0208 | +0.0378 | +0.0550 | +0.0714 |
| $\overline{p_3}$ | —0.3845 | —0.3520 | —0.3260 | —0.3032 | —0.2830 | —0.2635 | —0.2458 | —0.2290 | —0.2118 | —0.1952 | —0.1792 |
| $\overline{p_1^2}$ | 0.0762 | 0.0836 | 0.0899 | 0.0969 | 0.1040 | 0.1119 | 0.1206 | 0.1293 | 0.1398 | 0.1490 | 0.1596 |
| $\overline{p_2^2}$ | 0.0206 | 0.0150 | 0.0116 | 0.0092 | 0.0075 | 0.0065 | 0.0066 | 0.0067 | 0.0076 | 0.0092 | 0.0113 |
| $\overline{p_3^2}$ | 0.1520 | 0.1280 | 0.1101 | 0.0958 | 0.0840 | 0.0735 | 0.0641 | 0.0562 | 0.0488 | 0.0421 | 0.0362 |

TABLE I*d* ($\alpha_0 = 0.60$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.3240 | +0.3383 | +0.3495 | +0.3615 | +0.3733 | +0.3853 | +0.3980 | +0.4112 | +0.4248 | +0.4377 | +0.4505 |
| $\overline{p_2}$ | +0.0030 | +0.0240 | +0.0418 | +0.0594 | +0.0766 | +0.0944 | +0.1104 | +0.1254 | +0.1422 | +0.1584 | +0.1752 |
| $\overline{p_3}$ | —0.2368 | —0.2078 | —0.1850 | —0.1638 | —0.1435 | —0.1245 | —0.1072 | —0.0908 | —0.0735 | —0.0565 | —0.0395 |
| $\overline{p_1^2}$ | 0.1170 | 0.1258 | 0.1330 | 0.1410 | 0.1493 | 0.1582 | 0.1679 | 0.1783 | 0.1894 | 0.2002 | 0.2112 |
| $\overline{p_2^2}$ | 0.0059 | 0.0062 | 0.0072 | 0.0087 | 0.0111 | 0.0139 | 0.0168 | 0.0206 | 0.0249 | 0.0296 | 0.0354 |
| $\overline{p_2^2}$ | 0.0592 | 0.0462 | 0.0371 | 0.0298 | 0.0235 | 0.0185 | 0.0144 | 0.0109 | 0.0082 | 0.0059 | 0.0042 |

TABLE I*e* ($\alpha_0 = 0.50$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.4048 | +0.4172 | +0.4270 | +0.4370 | +0.4475 | +0.4585 | +0.4705 | +0.4827 | +0.4952 | +0.5065 | +0.5173 |
| $\overline{p_2}$ | +0.1226 | +0.1410 | +0.1560 | +0.1724 | +0.1872 | +0.2028 | +0.2172 | +0.2322 | +0.2484 | +0.2632 | +0.2804 |
| $\overline{p_3}$ | —0.0828 | —0.0595 | —0.0395 | —0.0210 | —0.0022 | +0.0162 | +0.0335 | +0.0482 | +0.0658 | +0.0822 | +0.0992 |
| $\overline{p_1^2}$ | 0.1731 | 0.1826 | 0.1905 | 0.1989 | 0.2080 | 0.2177 | 0.2287 | 0.2400 | 0.2518 | 0.2630 | 0.2738 |
| $\overline{p_2^2}$ | 0.0195 | 0.0242 | 0.0284 | 0.0337 | 0.0388 | 0.0449 | 0.0509 | 0.0573 | 0.0652 | 0.0727 | 0.0821 |
| $\overline{p_3^2}$ | 0.0094 | 0.0058 | 0.0039 | 0.0028 | 0.0020 | 0.0022 | 0.0032 | 0.0045 | 0.0062 | 0.0089 | 0.0118 |

TABLE II.

*Values of $p$.* (D-hyp.; larger star in front).

| $a'$ \ $k$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| 0.00.... | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 |
| 0.02.... | 0.796 | 0.800 | 0.804 | 0.808 | 0.812 | 0.816 | 0.820 | .823 | .827 |
| 0.05.... | .685 | .689 | .694 | .698 | .703 | .708 | .712 | .717 | .721 |
| 0.10.... | .543 | .550 | .558 | .564 | .569 | .576 | .583 | .591 | .599 |
| 0.15.... | .428 | .437 | .447 | .455 | .462 | .471 | .480 | .492 | .503 |
| 0.20.... | .328 | .338 | .349 | .359 | .369 | .379 | .389 | .402 | .415 |
| 0.25.... | .241 | .251 | .263 | .273 | .283 | .295 | .306 | .320 | .334 |
| 0.30.... | .160 | .169 | .181 | .192 | .203 | .216 | .229 | .244 | .259 |
| 0.35.... | .081 | .091 | .104 | .116 | .128 | .142 | .156 | .172 | .187 |
| 0.40.... | +0.004 | +0.017 | +0.032 | +0.045 | +0.057 | .072 | .085 | .101 | .117 |
| 0.45.... | —0.070 | —0.055 | —0.039 | —0.025 | —0.011 | +0.003 | +0.018 | +0.033 | +0.049 |
| 0.50.... | .143 | .126 | .108 | .093 | .078 | —0.064 | —0.049 | —0.034 | —0.018 |
| 0.55.... | .214 | .196 | .176 | .161 | .145 | .130 | .115 | .100 | .084 |
| 0.60.... | .284 | .265 | .245 | .229 | .213 | .197 | .182 | .167 | .151 |
| 0.65.... | .353 | .334 | .314 | .298 | .282 | .265 | .249 | .235 | .219 |
| 0.70.... | .424 | .403 | .383 | .367 | .351 | .334 | .318 | .304 | .290 |
| 0.75.... | .495 | .475 | .454 | .438 | .423 | .406 | .390 | .377 | .365 |
| 0.80.... | .570 | .549 | .528 | .513 | .498 | .482 | .467 | .454 | .443 |
| 0.85.... | .650 | .628 | .607 | .592 | .578 | .563 | .549 | .538 | .528 |
| 0.90.... | .737 | .716 | .697 | .682 | .668 | .656 | .644 | .635 | .626 |
| 0.95.... | .837 | .818 | .800 | .784 | .770 | .761 | .753 | .747 | .741 |
| 0.98.... | .910 | .889 | .876 | .865 | .855 | .850 | .846 | .840 | .836 |
| 0.99.... | —0.945 | 0.930 | 0.917 | 0.907 | 0.898 | 0.894 | 0.892 | 0.888 | 0.885 |
| 1.00.... | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 |
| $\overline{p_1}$.... | +0.0455 | +0.0578 | +0.0722 | +0.0847 | +0.0970 | +0.1107 | +0.1242 | +0.1393 | +0.1547 |
| $\overline{p_2}$.... | —0.3540 | —0.3346 | —0.3144 | —0.2986 | —0.2828 | —0.2664 | —0.2508 | —0.2366 | —0.2218 |
| $\overline{p_3}$.... | —0.6985 | —0.6778 | —0.6580 | —0.6428 | —0.6285 | —0.6155 | —0.6032 | —0.5935 | —0.5845 |
| $\overline{p_1^2}$.... | 0.0193 | 0.0198 | 0.0212 | 0.0227 | 0.0245 | 0.0272 | 0.0300 | 0.0340 | 0.0384 |
| $\overline{p_2^2}$.... | 0.1352 | 0.1216 | 0.1085 | 0.0987 | 0.0896 | 0.0805 | 0.0723 | 0.0655 | 0.0590 |
| $\overline{p_3^2}$.... | 0.4978 | 0.4694 | 0.4432 | 0.4234 | 0.4053 | 0.3897 | 0.3752 | 0.3642 | 0.3540 |

### TABLE II*a* ($\alpha_0' = 0.90$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1037 | +0.1148 | +0.1283 | +0.1402 | +0.1505 | +0.1650 | +0.1770 | +0.1918 | +0.2067 |
| $\overline{p_2}$ | —0.2624 | —0.2438 | —0.2246 | —0.2088 | —0.1932 | —0.1778 | —0.1624 | —0.1478 | —0.1318 |
| $\overline{p_3}$ | —0.5538 | —0.5328 | —0.5118 | —0.4962 | —0.4818 | —0.4655 | —0.4502 | —0.4380 | —0.4265 |
| $\overline{p_1^2}$ | 0.0254 | 0.0273 | 0.0304 | 0.0332 | 0.0361 | 0.0401 | 0.0442 | 0.0494 | 0.0553 |
| $\overline{p_2^2}$ | 0.0768 | 0.0672 | 0.0580 | 0.0512 | 0.0448 | 0.0390 | 0.0337 | 0.0293 | 0.0247 |
| $\overline{p_3^2}$ | 0.3122 | 0.2898 | 0.2678 | 0.2522 | 0.2380 | 0.2228 | 0.2088 | 0.1982 | 0.1886 |

### TABLE II*b* ($\alpha_0' = 0.80$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1627 | +0.1732 | +0.1858 | +0.1970 | +0.2080 | +0.2208 | +0.2332 | +0.2477 | +0.2623 |
| $\overline{p_2}$ | —0.1704 | —0.1536 | —0.1350 | —0.1202 | —0.1050 | —0.0900 | —0.0752 | —0.0600 | —0.0440 |
| $\overline{p_3}$ | —0.4242 | —0.4042 | —0.3838 | —0.3678 | —0.3522 | —0.3352 | —0.3195 | —0.3055 | —0.2920 |
| $\overline{p_1^2}$ | 0.0385 | 0.0420 | 0.0463 | 0.0502 | 0.0545 | 0.0597 | 0.0650 | 0.0718 | 0.0792 |
| $\overline{p_2^2}$ | 0.0356 | 0.0300 | 0.0243 | 0.0204 | 0.0168 | 0.0137 | 0.0113 | 0.0094 | 0.0077 |
| $\overline{p_3^2}$ | 0.1840 | 0.1676 | 0.1512 | 0.1392 | 0.1281 | 0.1165 | 0.1061 | 0.0975 | 0.0896 |

### TABLE II*c* ($\alpha_0' = 0.70$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.2247 | +0.2347 | +0.2463 | +0.2567 | +0.2672 | +0.2790 | +0.2905 | +0.3048 | +0.3188 |
| $\overline{p_2}$ | —0.0764 | —0.0616 | —0.0448 | —0.0310 | —0.0174 | —0.0030 | +0.0144 | +0.0268 | +0.0428 |
| $\overline{p_3}$ | —0.3010 | —0.2822 | —0.2622 | —0.2465 | —0.2305 | —0.2142 | —0.1988 | —0.1842 | —0.1682 |
| $\overline{p_1^2}$ | 0.0608 | 0.0652 | 0.0708 | 0.0758 | 0.0809 | 0.0872 | 0.0932 | 0.1017 | 0.1102 |
| $\overline{p_2^2}$ | 0.0112 | 0.0088 | 0.0068 | 0.0054 | 0.0047 | 0.0047 | 0.0044 | 0.0053 | 0.0064 |
| $\overline{p_3^2}$ | 0.0938 | 0.0826 | 0.0718 | 0.0638 | 0.0560 | 0.0489 | 0.0422 | 0.0369 | 0.0314 |

TABLE II$d$ ($\alpha'_0 = 0.60$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.2890 | +0.2985 | +0.3097 | +0.3195 | +0.3290 | +0.3400 | +0.3507 | +0.3643 | +0.3775 |
| $\overline{p_2}$ | +0.0204 | +0.0326 | +0.0472 | +0.0598 | +0.0724 | +0.0866 | +0.1004 | +0.1162 | +0.1318 |
| $\overline{p_3}$ | —0.1782 | —0.1610 | —0.1420 | —0.1270 | —0.1115 | —0.0970 | —0.0820 | —0.0670 | —0.0510 |
| $\overline{p_1^2}$ | 0.0920 | 0.0975 | 0.1042 | 0.1102 | 0.1161 | 0.1232 | 0.1303 | 0.1399 | 0.1494 |
| $\overline{p_2^2}$ | 0.0046 | 0.0050 | 0.0059 | 0.0072 | 0.0088 | 0.0109 | 0.0134 | 0.0170 | 0.0209 |
| $\overline{p_3^2}$ | 0.0341 | 0.0282 | 0.0224 | 0.0181 | 0.0145 | 0.0115 | 0.0088 | 0.0064 | 0.0045 |

TABLE II$e$ ($\alpha'_0 = 0.50$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|---|---|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.3575 | +0.3670 | +0.3775 | +0.3867 | +0.3953 | +0.4052 | +0.4148 | +0.4273 | +0.4393 |
| $\overline{p_2}$ | +0.1206 | +0.1308 | +0.1434 | +0.1548 | +0.1664 | +0.1800 | +0.1930 | +0.2084 | +0.2234 |
| $\overline{p_3}$ | —0.0512 | —0.0368 | —0.0212 | —0.0072 | +0.0062 | +0.0208 | +0.0348 | +0.0502 | +0.0662 |
| $\overline{p_1^2}$ | 0.1348 | 0.1415 | 0.1492 | 0.1561 | 0.1624 | 0.1703 | 0.1779 | 0.1884 | 0.1986 |
| $\overline{p_2^2}$ | 0.0178 | 0.0201 | 0.0236 | 0.0270 | 0.0304 | 0.0352 | 0.0400 | 0.0460 | 0.0526 |
| $\overline{p_3^2}$ | 0.0042 | 0.0028 | 0.0020 | 0.0016 | 0.0014 | 0.0019 | 0.0025 | 0.0039 | 0.0059 |

TABLE III.

*Values of p.* (D-hyp. ; smaller star in front).

| α″ \ k | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| 0.00 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 | +1.000 |
| 0.02 | .796 | .786 | .778 | .772 | .768 | .765 | .762 |
| 0.05 | .685 | .660 | .644 | .635 | .628 | .624 | .620 |
| 0.10 | .543 | .511 | .495 | .485 | .479 | .473 | .468 |
| 0.15 | .428 | .390 | .370 | .357 | .350 | .344 | .338 |
| 0.20 | .328 | .287 | .263 | .250 | .240 | .232 | .226 |
| 0.25 | .240 | .195 | .168 | .150 | .142 | .134 | .127 |
| 0.30 | .158 | .109 | +0.079 | +0.060 | +0.049 | +0.042 | +0.036 |
| 0.35 | .080 | +0.029 | —0.003 | —0.024 | —0.036 | —0.043 | —0.048 |
| 0.40 | +0.004 | —0.048 | .083 | .104 | .117 | .124 | .129 |
| 0.45 | —0.070 | .125 | .162 | .182 | .196 | .204 | .209 |
| 0.50 | .143 | .200 | .237 | .257 | .272 | .280 | .285 |
| 0.55 | .214 | .271 | .308 | .329 | .343 | .351 | .356 |
| 0.60 | .284 | .341 | .379 | .401 | .413 | .421 | .427 |
| 0.65 | .354 | .411 | .450 | .472 | .483 | .491 | .497 |
| 0.70 | .424 | .482 | .521 | .542 | .553 | .561 | .567 |
| 0.75 | .496 | .554 | .592 | .612 | .623 | .631 | .637 |
| 0.80 | .570 | .627 | .663 | .682 | .693 | .700 | .706 |
| 0.85 | .650 | .704 | .736 | .754 | .763 | .770 | .776 |
| 0.90 | .737 | .787 | .814 | .828 | .835 | .842 | .847 |
| 0.95 | .837 | .880 | .899 | .907 | .913 | .918 | .922 |
| 0.98 | .910 | .943 | .955 | .959 | .962 | .964 | .967 |
| 1.00 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 | —1.000 |
| 1+0.2x | —1.000 | —1.014 | —1.030 | —1.047 | —1.070 | —1.100 | —1.145 |
| 1+0.4x | 1.000 | 1.030 | 1.062 | 1.099 | 1.147 | 1.216 | 1.312 |
| 1+0.6x | 1.000 | 1.048 | 1.100 | 1.163 | 1.241 | 1.355 | 1.516 |
| 1+0.8x | 1.000 | 1.069 | 1.146 | 1.246 | 1.361 | 1.533 | 1.780 |
| 1+x | —1.000 | —1.111 | —1.250 | —1.429 | —1.667 | —2.000 | —2.500 |
| x | 0.000 | 0.015 | 0.047 | 0.084 | 0.143 | 0.220 | 0.322 |

Table III (Continued).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.0448 | —0.0067 | —0.0395 | —0.0595 | —0.0717 | —0.0792 | —0.0847 |
| $\overline{p_2}$ | —0.3542 | —0.4118 | —0.4500 | —0.4712 | —0.4830 | —0.4908 | —0.4968 |
| $\overline{p_3}$ | —0.6985 | —0.7495 | —0.7780 | —0.7928 | —0.8010 | —0.8075 | —0.8128 |
| $\overline{p_1^2}$ | 0.0190 | 0.0181 | 0.0206 | 0.0228 | 0.0250 | 0.0262 | 0.0269 |
| $\overline{p_1^2}$ | 0.1353 | 0.1796 | 0.2126 | 0.2320 | 0.2431 | 0.2506 | 0.2565 |
| $\overline{p_1^2}$ | 0.4978 | 0.5706 | 0.6130 | 0.6354 | 0.6483 | 0.6586 | 0.6670 |

Table III*a* ($\alpha_0'' = 0.90$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1028 | +0.0533 | +0.0215 | +0.0025 | —0.0088 | —0.0163 | —0.0220 |
| $\overline{p_2}$ | —0.2628 | —0.3198 | —0.3580 | —0.3788 | —0.3916 | —0.4000 | —0.4052 |
| $\overline{p_3}$ | —0.5540 | —0.5852 | —0.6460 | —0.6650 | —0.6755 | —0.6830 | —0.6890 |
| $\overline{p_1^2}$ | 0.0251 | 0.0184 | 0.0169 | 0.0169 | 0.0175 | 0.0176 | 0.0177 |
| $\overline{p_1^2}$ | 0.0791 | 0.1102 | 0.1364 | 0.1517 | 0.1614 | 0.1677 | 0.1723 |
| $\overline{p_1^2}$ | 0.3103 | 0.3780 | 0.4226 | 0.4473 | 0.4612 | 0.4714 | 0.4796 |

Table III*b* ($\alpha_0'' = 0.80$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.1622 | +0.1143 | +0.0848 | +0.0663 | +0.0555 | +0.0483 | +0.0423 |
| $\overline{p_2}$ | —0.1704 | —0.2268 | —0.2638 | —0.2846 | —0.2984 | —0.3064 | —0.3116 |
| $\overline{p_3}$ | —0.4250 | —0.4828 | —0.5210 | —0.5420 | —0.5530 | —0.5610 | —0.5670 |
| $\overline{p_1^2}$ | 0.0385 | 0.0261 | 0.0211 | 0.0190 | 0.0180 | 0.0171 | 0.0164 |
| $\overline{p_2^2}$ | 0.0336 | 0.0581 | 0.0763 | 0.0879 | 0.0958 | 0.1006 | 0.1039 |
| $\overline{p_3^2}$ | 0.1847 | 0.2372 | 0.2754 | 0.2977 | 0.3098 | 0.3186 | 0.3254 |

TABLE IIIc ($\alpha_0'' = 0.70$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.2237 | +0.1778 | +0.1503 | +0.1332 | +0.1232 | +0.1158 | +0.1098 |
| $\overline{p_2}$ | —0.0762 | —0.1316 | —0.1678 | —0.1878 | —0.2022 | —0.2096 | —0.2146 |
| $\overline{p_3}$ | —0.3015 | —0.3585 | —0.3968 | —0.4185 | —0.4305 | —0.4385 | —0.4442 |
| $\overline{p_1^2}$ | 0.0602 | 0.0427 | 0.0344 | 0.0301 | 0.0278 | 0.0260 | 0.0246 |
| $\overline{p_2^2}$ | 0.0111 | 0.0229 | 0.0339 | 0.0389 | 0.0467 | 0.0498 | 0.0519 |
| $\overline{p_3^2}$ | 0.0939 | 0.1315 | 0.1605 | 0.1782 | 0.1883 | 0.1953 | 0.2009 |

TABLE IIId ($\alpha_0'' = 0.60$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.2880 | +0.2448 | +0.2195 | +0.2038 | +0.1948 | +0.1875 | +0.1813 |
| $\overline{p_2}$ | +0.0198 | —0.0324 | —0.0666 | —0.0870 | —0.1000 | —0.1072 | —0.1122 |
| $\overline{p_3}$ | —0.1782 | —0.2350 | —0.2720 | —0.2928 | —0.3070 | —0.3150 | —0.3200 |
| $\overline{p_1^2}$ | 0.0914 | 0.0691 | 0.0580 | 0.0518 | 0.0484 | 0.0457 | 0.0434 |
| $\overline{p_2^2}$ | 0.0045 | 0.0054 | 0.0090 | 0.0121 | 0.0147 | 0.0162 | 0.0173 |
| $\overline{p_3^2}$ | 0.0340 | 0.0575 | 0.0763 | 0.0881 | 0.0966 | 0.1015 | 0.1047 |

TABLE IIIe ($\alpha_0'' = 0.50$).

| $k =$ | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| $\overline{p_1}$ | +0.3568 | +0.3165 | +0.2937 | +0.2797 | +0.2717 | +0.2645 | +0.2582 |
| $\overline{p_2}$ | +0.1196 | +0.0698 | +0.0388 | +0.0190 | +0.0074 | +0.0004 | —0.0052 |
| $\overline{p_3}$ | —0.0515 | —0.1058 | —0.1420 | —0.1622 | —0.1760 | —0.1835 | —0.1885 |
| $\overline{p_1^2}$ | 0.1342 | 0.1076 | 0.0942 | 0.0865 | 0.0823 | 0.0786 | 0.0754 |
| $\overline{p_2^2}$ | 0.0174 | 0.0081 | 0.0049 | 0.0039 | 0.0037 | 0.0037 | 0.0036 |
| $\overline{p_3^2}$ | 0.0044 | 0.0130 | 0.0220 | 0.0257 | 0.0329 | 0.0356 | 0.0375 |

# STELLINGEN

TABLE III [illegible]

| [illegible] | 1 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| [illegible] | [illegible] | +0.1379 | +0.1262 | +0.1242 | +0.1208 | +0.1166 | +0.1098 |
| [illegible] | −0.0768 | −0.1318 | −0.1628 | [illegible] | −0.2063 | −0.2099 | −0.2146 |
| [illegible] | −0.3010 | −0.3383 | −0.3914 | −0.4103 | −0.4320 | −0.4398 | −0.4442 |
| [illegible] | 0.0009 | 0.0498 | 0.0841 | [illegible] | 0.0978 | [illegible] | 0.0240 |
| [illegible] | 0.0111 | 0.0925 | 0.0329 | 0.1099 | 0.0407 | [illegible] | 0.0510 |
| [illegible] | 0.1009 | 0.1317 | 0.1609 | 0.1709 | 0.1581 | [illegible] | 0.2069 |

TABLE III [illegible]

| [illegible] | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| [illegible] | [illegible] | +0.0800 | +0.0198 | [illegible] | +0.1308 | [illegible] | +0.1615 |
| [illegible] | [illegible] | [illegible] | −0.0998 | [illegible] | [illegible] | [illegible] | −0.1192 |
| [illegible] | [illegible] | [illegible] | −0.0750 | [illegible] | −0.3012 | −0.3128 | −0.3290 |
| [illegible] | [illegible] | [illegible] | 0.0280 | [illegible] | 0.0084 | 0.0205 | 0.0432 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.0847 | 0.0549 | 0.0173 |
| [illegible] | 0.0140 | [illegible] | [illegible] | [illegible] | 0.0909 | 0.0918 | 0.1047 |

TABLE III [illegible]

| [illegible] | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 |
|---|---|---|---|---|---|---|---|
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | +0.0646 | +0.0902 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | −0.0904 | −0.0970 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | −0.1695 | −0.1880 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.0788 | 0.0764 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.0697 | 0.0690 |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.0484 | 0.0376 |

# STELLINGEN

# STELLINGEN.

I

De waarnemingen en theorieën van SHAPLEY leveren geen grond voor de opvatting dat er geen selectieve absorptie van het licht in de ruimte zou zijn.

II

Zonder de kennis van de absolute helderheid van eenige kort-periodische Cepheïden mist de kromme, die het verband tusschen lichtkracht en periode voor deze sterren aangeeft, een vasten grondslag.

III

De afleiding door SHAPLEY van de afstanden der bolvormige sterrenhoopen berust op onjuisten grondslag.

H. SHAPLEY: „Studies based on the Colors and Magnitudes in Stellar Clusters". (Contr. Mount Wilson Sol. Obs. VI—X.)

IV

Tegen de opvatting van een Cepheïde als een pulseerende ster zijn verschillende bedenkingen aan te voeren.

V

Nauwkeurig bepaalde lichtkrommen van sommige Algol- (of β Lyrae-) veranderlijken kunnen een oordeel geven omtrent de juistheid van de gebruikte photometrische schaal.

VI

Lichtkrommen afgeleid met behulp van graadwaarnemingen, behoeven niet onder te doen voor lichtkrommen afgeleid uit photometrische bepalingen.

## VII

Tegen PEROT's opvatting, dat hij het EINSTEIN-effect geconstateerd heeft voor één der Magnesium-lijnen in het zonnespectrum, bestaan ernstige bedenkingen.

A. PÉROT: Journal de Physique, April 1922. C. R., Maart 1921.

## VIII

In de „fijnstructuur" van het Helium-spectrum mag men geen bewijs zien van de juistheid der Relativiteitstheorie.

A. SOMMERFELD: „Atombau und Spectrallinien", 1922 (8. Kap. § 7).

## IX

De hypothese van SILBERSTEIN, dat de twee electronen in het Helium-atoom elkander niet zouden afstooten, is eerst dan gerechtvaardigd, als men die afstooting op grond van de gebruikelijke quantentheoretische eigenschappen van het He-atoom niet kan verklaren.

De goede overeenstemming van de meeste der waargenomen spectraallijnen met lijnen, die volgens de hypothese van SILBERSTEIN mogelijk zijn, kan a priori verwacht worden.

L. SILBERSTEIN: Ap. J. September 1922.

## X

De serieformule van MARSHALL WATTS heeft geen beteekenis. Ook die van RAMAGE bevat te veel constanten.

Phil. Mag. 18, 411 (1909).
Proc. Royal Soc. 70, 1 en 303 (1902).

## XI

In tegenstelling met EINSTEIN komt BECQUEREL tot de gevolgtrekking, dat een waarnemer in het middelpunt van een draaiende schijf de verhouding van den cirkelomtrek tot de middellijn $< \pi$ vindt.

Deze gevolgtrekking is aannemelijker.

M. J. BECQUEREL: „Le Principe de la Relativité et la Théorie de la Gravitation," 1922. Errata et Additions p. VIII.

## XII

De vermindering van intensiteit, die een lichtbundel ondergaat bij zijn doorgang door een laagje water, moet, buiten het gebied der absorptiebanden, waarschijnlijk grootendeels niet aan ware absorptie worden toegeschreven.

THOS. EWAN: Proc. Royal Soc. 57, 126 (1894).
E. ASCHKINASS: Wied. Ann. der Physik und Chemie 55, 401 (1895).

## XIII

De oplossing van een differentiaalvergelijking van den vorm

$$\frac{d^n y}{dx^n} + a\,\frac{d^{n-1}y}{dx^{n-1}} + b\,\frac{d^{n-2}y}{dx^{n-2}} + \cdot\ \cdot\ \cdot\ \cdot\ \cdot + p\,\frac{dy}{dx} + qy = 0$$

voert volgens de bekende behandelingswijze tot de $u$-vergelijking:

$$u^n + a\,u^{n-1} + b\,u^{n-2} + \cdot\ \cdot\ \cdot\ \cdot\ \cdot + p\,u + q = 0$$

Heeft deze gelijke wortels dan bestaat er een methode die vlugger en eenvoudiger tot de algemeene integraal voert dan de drie gebruikelijke methoden.

## XIV

De juistheid van de formule die Paul Hertz afleidt voor het aequivalentgeleidingsvermogen mag, voor het geval van zeer groote en van oneindige verdunning ($\Lambda_\infty$), door de toetsing van Lorenz niet als bewezen worden beschouwd.

Richard Lorenz: Zeitschrift für Anorg. Chem. Bd. 114 und 116.